From Kalman to Hodrick-Prescott filter

Theory and Application

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Outline

Beginnings of the Kalman Filter

- Rudolph E. Kalman (1960) published a paper in the *Journal of Basic Engineering* describing a recursive solution to the discrete-data linear filtering problem.
- Its usage has spread into a tremendously broad range of areas, such as nuclear medicine, chemistry, satellite orbit estimation, statistics, finance, econometrics, etc.
- Produces an estimate of some unobservable system state x_t at time t, using the measurements $y_0, y_1, ..., y_t$ of some observable variable y_t.
- Traditional AR, MA, and ARMA models versus optimal control theory way of representing dynamic phenomena through state space models.

Kalman Filter in Economics Literature

Macroeconomics

- Chen (2001): real interest rates, inflation, and the term structure of interest rate under the expectations hypothesis.
- Harvey and Timbur (2003):trends and cycles in U.S. investment and GDP.
- Koopman and Bos (2004): state space model and the stochastic volatility model, modeling and forecasting using U.S. monthly inflation rates.
- Kuttner (1994): estimating potential output, potential real GDP is modeled as an unobserved stochastic trend.
- Laubach and Williams (2003): time-varying natural rates of interest and output and trend growth.
- Burmeister and Wall (1982): German hyperinflation, rational expectations.

Kalman Filter in Economics Literature

Microeconomics

- Tegene (1990, 1991): estimating price, income and advertising elasticities in order to examine structural changes in the demand for alcoholic beverages and cigarettes in the U.S.
- Engle and Watson (1981): consider the unobserved factors underlying metropolitan wage rate for Los Angeles, based on observations of sectoral wages within the Standard Metropolitan Statistical Area.

Motivation for this paper

Hodrick and Prescott (1997): "Postwar U.S. Business Cycles: An Empirical Investigation"

- use the Kalman filter to develop their own so-called HP filter,
- propose a procedure for representing a time series as the sum of a smoothly varying trend component and a cyclical component.
- use several macroeconomic time series - GNP, inflation, unemployment rate.

In this paper, I analyze U.S. GDP quarterly data using their proposed method.

State Space Representation

Let an $(l \times 1)$ vector y_t be the observed random variable at time t. Let an $(n \times 1)$ vector x_t be the unobserved random variable at time t, which is a theoretical construct, and we call it the state vector. Let the state disturbance be represented by an $(N \times 1)$ vector w_t, and the measurement error by an $(l \times 1)$ vector e_t.

$$x_{t+1} = Ax_t + w_t \tag{1}$$

$$y_t = Cx_t + e_t. \tag{2}$$

Noise Processes Assumptions

$$E\left(w_t e_\tau^{'}\right) = 0 \qquad \text{for all } t, \tau. \tag{3}$$

$$E\left(w_t w_\tau^{'}\right) = \begin{cases} Q & \text{for } t = \tau \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

$$E\left(e_t e_\tau^{'}\right) = \begin{cases} R & \text{for } t = \tau \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Initial Value of the State Vector Assumptions

$$E\left(w_t x_t^{'}\right) = 0 \qquad \text{for all } t = 1, 2, ..., T, \tag{6}$$

$$E\left(e_t x_t^{'}\right) = 0 \qquad \text{for all } t = 1, 2, ..., T. \tag{7}$$

The initial condition is unobserved, but it is known that x_0 is Gaussian with mean $\hat{x}_0$ and variance Σ_0.

Discrete Kalman Filter Algorithm

Time Update Equations

$$x_{t|t-1} = Ax_{t-1|t-1} \tag{8}$$

$$\Sigma_{t|t-1} = A\Sigma_{t-1|t-1}A^T + Q \tag{9}$$

Measurement Update Equations

$$K_t = \Sigma_{t|t-1}C(C^T\Sigma_{t|t-1}C + R)^{-1} \tag{10}$$

$$\hat{x}_{t|t} = x_{t|t-1} + K_t(y_t - C^T x_{t|t-1}) \tag{11}$$

$$\Sigma_{t|t} = (I - K_tC)\Sigma_{t|t-1} \tag{12}$$

Example

Let $A = 0.9$, $C = 1$, $\Sigma_0 = 40,000$, $x_0 = 1000$, $Q = 100$, $R = 10,000$, and we observe $y_1 = 1200$.

The results from the time update equations are:
$x_{1|0} = Ax_0 = 0.9 * 1000 = 900$,

$\Sigma_{1|0} = 0.9^2 * 40,000 + 100 = 32,500$.

The results from the measurement update equations are:
$K_1 = \Sigma_{1|0} C (C^2 \Sigma_{1|0} + R)^{-1} = 0.7647$,

$\hat{x}_{1|1} = x_{1|0} + K_1 (y_1 - C x_{1|0}) = 1129$,

$\Sigma_{1|1} = \Sigma_{1|0} (1 - K_1 C) + R K_1^2 = 7647$.

HP Filter

The HP filter decomposes y_t into a time trend or growth component, g_t, and a stationary residual or cyclical component ,c_t. Then the series can be modeled as follows:

$$y_t = g_t + c_t, \tag{13}$$

where the unobservable growth component g is

$$g_t = 2g_{t-1} - g_{t-2} + \epsilon_t. \tag{14}$$

HP Filter State Space Representation

$$x_t = \begin{pmatrix} g_t \\ g_{t-1} \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad w_t = \begin{pmatrix} \epsilon_t \\ 0 \end{pmatrix}.$$

HP Filter

The simple HP filter gives an estimate of the unobserved variable as the solution to the following minimization problem:

$$Min \sum_{t=1}^{T} (y_t - g_t)^2 + \lambda \sum_{t=2}^{T-1} [(g_{t+1} - g_t) - (g_t - g_{t-1})]^2, \qquad (15)$$

with respect to $\{g_t\}_{t=1}^{T}$.

The ratio of variances between c_t and ϵ_t is λ in the HP filter function. We take λ = 1600.

Unit Root Tests

Examine three models:

$$\Delta y_t = \delta y_{t-1} + v_t, \tag{16}$$

such that y_t is a random walk,

$$\Delta y_t = \beta_1 + \delta y_{t-1} + v_t, \tag{17}$$

such that y_t is a random walk with drift, and

$$\Delta y_t = \beta_1 + \beta_2 t + \delta y_{t-1} + v_t, \tag{18}$$

such that y_t is a random walk with drift around a stochastic trend.

The hypothesis being tested is whether $\delta = 0$, the time series is nonstationary, against the alternative $\delta < 0$, the time series is stationary.

Unit Root Tests

ADF t statistic	Critical Value at 5%
5.810	-1.944
-0.135	-2.895
-1.642	-3.462

Table: Unit Root Test for GDP

Therefore, the data is nonstationary, or in other words, it has a unit root.

GDP as Sum of Trend Component and Cyclical Component

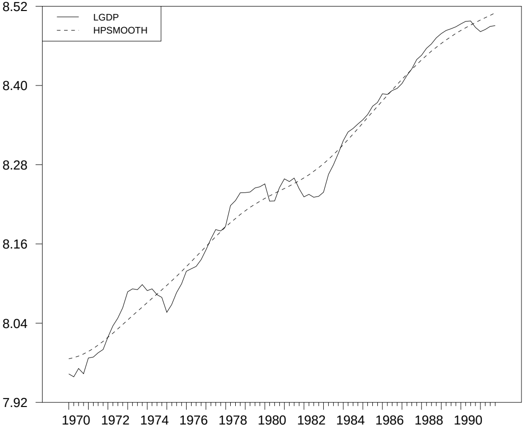

Figure: 1

The Cyclical Component of GDP

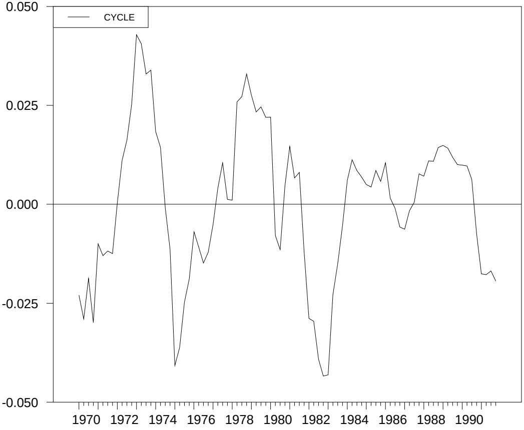

Figure: 2

Conclusion

- Present a theoretical overview of the discrete Kalman filter.
- Test the given theoretical framework empirically.
- The Hodrick-Prescott (1997) filter was rewritten in a state space form, and it was estimated by maximum likelihood method with the Kalman filter.
- The nature of the comovements of the cyclical component of the U.S. GDP time series is very different from the comovements of the slowly varying trend component.

Extensions

Possible extensions include the following:

- Check how different paramter values of λ would change the results of the estimation.
- Examine the validity of the model, and the serial correlation properties of the data.
- Examine how well the Kalman filter could forecast trends and cycles based on the GDP data that was used in the paper.

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