Workouts in Hypothesis Testing: Part II

6. y is a Bernoulli random variable, a single paramter distribution. P(y=1) = θ and P(y =0) = 1-θ. To keep things simple suppose there are 5 trials.

The log likelihood is

$$\left[\sum_{t=1}^{5}\left[y_t\cdot\ln(\theta)+\left(1-y_t\right)\cdot\ln(1-\theta)\right]\right]$$

The first order condition for a maximum is to take the derivative with respect to θ and set the result equal to zero.

$$\frac{d}{d\theta}\left[\sum_{t=1}^{5}\left[y_t\cdot\ln(\theta)+\left(1-y_t\right)\cdot\ln(1-\theta)\right]\right]$$

This derivative is

$$\frac{y_1}{\theta}-\frac{\left(1-y_1\right)}{(1-\theta)}+\frac{y_2}{\theta}-\frac{\left(1-y_2\right)}{(1-\theta)}+\frac{y_3}{\theta}-\frac{\left(1-y_3\right)}{(1-\theta)}+\frac{y_4}{\theta}-\frac{\left(1-y_4\right)}{(1-\theta)}+\frac{y_5}{\theta}-\frac{\left(1-y_5\right)}{(1-\theta)}$$

Setting this result equal to zero and solving for θ.

$$\theta= \frac{1}{5}\cdot y_1+\frac{1}{5}\cdot y_2+\frac{1}{5}\cdot y_3+\frac{1}{5}\cdot y_4+\frac{1}{5}\cdot y_5$$

The maximum likelihood estimator for θ is the sample proportion.

We wish to test the hypothesis

H_o: $\theta_o := .4$

H_1: θ_o <>.4

Our sample results are $n := 100$ and $success := 65$ =(sum of y_t).

So $phat := .65$

6. a. Likelihood Ratio Test

The log-likelihood function evaluated at H_o is

$$L_o := (\text{success})\cdot\ln(\theta_o) + (n - \text{success})\cdot\ln(1 - \theta_o)$$

$$L_o = -77.438$$

The likelihood evaluated at the sample is

$$L_1 := \text{success}\cdot\ln(\text{phat}) + (n - \text{success})\cdot\ln(1 - \text{phat})$$

$$L_1 = -64.745$$

$$LR := -2\cdot(L_o - L_1)$$

$$LR = 25.386$$

The LR is asymptotically χ^2. In this case it has one degree of freedom. At the 5% level the critical value is 3.84, so we reject the null.

6.b. Wald Test

Asymptotically the sample proportion is $N(\theta, \theta(1-\theta)/n)$

$$W := (\text{phat} - \theta_o)\cdot\left[\text{phat}\cdot\frac{(1 - \text{phat})}{n}\right]^{-1}\cdot(\text{phat} - \theta_o)$$

$$W = 27.473$$

Since W is also χ^2 with one degree of freedom, the null hypothesis is rejected.

6.c. Lagrange Multiplier Test

See Greene, pp. 171

$$LM := -\left[\left(\frac{\text{success}}{\theta_o} - \frac{n - \text{success}}{1 - \theta_o}\right)\cdot\left[\frac{-\text{success}}{\theta_o^{\,2}} - \frac{(n - \text{success})}{(1 - \theta_o)^2}\right]^{-1}\cdot\left(\frac{\text{success}}{\theta_o} - \frac{n - \text{success}}{1 - \theta_o}\right)\right]$$

The first and third terms are the derivative of the log likelihood evaluated at the null hypothesis. The middle term is the inverse of the information matrix (the minus sign is out front), the variance of the first derivative.

$LM = 21.552$

The LM statistic is χ^2 with one degree of freedom, so once again we reject the null hypothesis.

7. The Normal Distribution: A 2 parameter distribution

The weight, x, in ounces, of a 10 lbs bag of sugar is N(μ,5). We wish to test the hypothesis

H_o: $\mu := 162$
H_1: μ<>162

The unrestricted parameter space is $\Omega=\{\mu, \sigma^2=5 :-\infty<\mu<\infty\}$. The restricted parameter space is $\omega=\{\mu=162, \sigma^2=5\}$. Suppose that for this example we have observed

$$x := \begin{bmatrix} 159 \\ 161 \\ 155 \\ 165 \\ 161 \end{bmatrix}$$

$n := 5$

$\sigma := \sqrt{5}$

$xbar := mean(x)$

$xbar = 160.2$

7.a. The Likelihood Ratio Test

The restricted likelihood is

$$L\omega := \left(\frac{1}{2\cdot\pi\cdot\sigma^2}\right)^{\frac{n}{2}} \cdot e^{\frac{-1}{2\cdot\sigma^2}\cdot\left[\sum_{i=1}^{n}\left(x_i-\mu\right)^2\right]}$$

$L\omega = 1.822 \cdot 10^{-7}$

The unrestricted likelihood is

$$L\Omega := \left(\frac{1}{2\cdot\pi\cdot\sigma^2}\right)^{\frac{n}{2}} \cdot e^{\frac{-1}{2\cdot\sigma^2}\cdot\left[\sum_{i=1}^{n}\left(x_i-xbar\right)^2\right]}$$

$$L\Omega = 9.206 \cdot 10^{-7}$$

The likelihood ratio is

$$\lambda := \frac{L\omega}{L\Omega} \qquad \lambda = 0.198$$

and the test statistic is

$$LR := -2 \cdot \ln(\lambda) \qquad LR = 3.24$$

A χ^2 with one degree of freedom at the 5% level is 3.84, so we fail to reject the null.

7.b. The Wald Test

$$W := \frac{(xbar - \mu)^2}{\frac{\sigma^2}{n}}$$

Ordinarlily we would use s^2, the unrestricted estimate of the variance, instead of σ^2 in the denominator. But in this example we know the population variance, so it makes no sense to estimate it.

$$W = 3.24$$

Again, we fail to reject the null

7.c. The Lagrange Multiplier Test

The log likelihood is

$$\ln\left[\left(\frac{1}{2 \cdot \pi \cdot \sigma^2}\right)^{\frac{n}{2}} \cdot e^{\frac{-1}{2 \cdot \sigma^2} \cdot \sum_{i=1}^{5} (x_i - \mu)^2}\right]$$

The derivative of the log likelihood with respect to the unknown parameter is

$$\frac{-1}{(2 \cdot \sigma^2)} \cdot (-2 \cdot x_1 + 10 \cdot \mu - 2 \cdot x_2 - 2 \cdot x_3 - 2 \cdot x_4 - 2 \cdot x_5)$$

The second derivative is $\frac{-5}{\sigma^2}$

The negative of the inverse of this is the variance of the the first derivative. The LM statistic is

$$LM := \left[\frac{-1}{\left(2\cdot\sigma^2\right)}\cdot\left(-2\cdot x_1 + 10\cdot\mu - 2\cdot x_2 - 2\cdot x_3 - 2\cdot x_4 - 2\cdot x_5\right)\right]^2\cdot\left[-\left(\frac{-5}{\sigma^2}\right)\right]^{-1}$$

$$LM = 3.24$$

The null is again rejected.

8. The Normal again, but with both parameters unknown.

H_o: μ=162
H_1: μ<>162

The restricted and unrestricted parameter spaces are
$\omega = \{(\mu,\sigma^2): \mu=162,\ 0<\sigma^2<\infty\}$ and
$\Omega = \{(\mu,\sigma^2): -\infty<\mu<\infty,\ 0<\sigma^2<\infty\}$

The ML estimates of the unknown parameters are

$$xbar = 160.2$$

$$s := \sqrt{\frac{1}{n}\cdot\sum_{i=1}^{5}\left(x_i - xbar\right)^2} \qquad s^2 = 10.56$$

The unrestricted likelihood, evaluated at the sample, is

$$L\Omega := \left(\frac{1}{2\cdot\pi\cdot s^2}\right)^{\frac{n}{2}}\cdot e^{\frac{-1}{2\cdot s^2}\cdot\left[\sum_{i=1}^{n}\left(x_i - xbar\right)^2\right]}$$

$$L\Omega = 2.289\bullet 10^{-6}$$

The restricted likelihood is

$$L\omega := \left(\frac{1}{2\cdot\pi\cdot s^2}\right)^{\frac{n}{2}}\cdot e^{\frac{-1}{2\cdot s^2}\cdot\left[\sum_{i=1}^{n}\left(x_i - 162\right)^2\right]}$$

$$L\omega = 1.063\bullet 10^{-6}$$

Their ratio is

$$\lambda := \frac{L\omega}{L\Omega}$$

$$\lambda = 0.464$$

And the test statistic is

$$LR := -2\cdot\ln(\lambda)$$

$$LR = 1.534$$

Do not reject the null.

8.b. The Wald Test

$$W := \frac{(xbar - \mu)^2}{\left(\frac{s^2}{n}\right)}$$

$$W = 1.534$$

Do not reject the null.

8.c. The Lagrange Multiplier Test

The log likelihood is

$$\frac{-n}{2}\cdot\ln(2\cdot\pi) - \frac{n}{2}\cdot\ln\left(\sigma^2\right) - \frac{1}{2\cdot\sigma^2}\cdot\sum_{i=1}^{n}\left(\left(x_i - \mu\right)\right)^2$$

We need the first derivatives in order to compute the efficient score or gradient vector. The derivative with respect to σ^2 is

$$\frac{d}{d\sigma 2}\left[\frac{-n}{2}\cdot\ln(2\cdot\pi) - \frac{n}{2}\cdot\ln(z) - \frac{1}{2\cdot\sigma^2}\cdot\sum_{i=1}^{5}\left(\left(x_i - \mu\right)\right)^2\right]$$

After some manipulation

$$\frac{-n}{2\cdot\sigma^2} + \frac{1}{2\cdot\left(\sigma^2\right)^2}\cdot\sum_{i=1}^{n}\left(x_i - \mu\right)^2$$

The derivative with respect to μ is

$$\frac{d}{d\mu}\left[\frac{-n}{2}\cdot\ln(2\cdot\pi) - \frac{n}{2}\cdot\ln\left(\sigma^2\right) - \frac{1}{2}\cdot\sum_{i=1}^{5}\left[\frac{\left(x_i - \mu\right)}{\sigma}\right]^2\right]$$

After some manipulation

$$\frac{1}{\sigma^2}\cdot\left[\sum_{i=1}^{n}\left(x_i - \mu\right)\right]$$

The Covariance matrix for the ML estimators is $\begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\cdot\sigma^4}{n} \end{bmatrix}$

We evaluate σ^2 under the restriction implicit in the null:

$$\sigma := \sqrt{\frac{1}{n}\cdot\sum_{i=1}^{n}\left(x_i-\mu\right)^2}$$

The LM statistic is

$$LM := \left[\frac{1}{\sigma^2}\cdot\left[\sum_{i=1}^{n}\left(x_i-\mu\right)\right] \quad \frac{-n}{2\cdot\sigma^2}+\frac{1}{2\cdot\left(\sigma^2\right)^2}\cdot\sum_{i=1}^{n}\left(x_i-\mu\right)^2\right]\cdot\begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\cdot\sigma^4}{n} \end{bmatrix}\cdot\begin{bmatrix} \frac{1}{\sigma^2}\cdot\left[\sum_{i=1}^{n}\left(x_i-\mu\right)\right] \\ \frac{-n}{2\cdot\sigma^2}+\frac{1}{2\cdot\left(\sigma^2\right)^2}\cdot\sum_{i=1}^{n}\left(x_i-\mu\right)^2 \end{bmatrix}$$

$LM = 1.174$

We cannot reject the null.