Hypothesis Testing Workouts: Part III

9. The Exponential Distribution

Greene, Econometric Analysis, 3rd Edition, P. 167

$n := 20 \qquad i := 1 .. n$

We have a restricted model of the relationship between income (y) and education (x).

$$f\omega(y,x,\beta) := \frac{1}{\beta + x_i} \cdot e^{\frac{-y_i}{\beta + x_i}}$$

And an unrestricted model

$$f\Omega(y,x,\beta,\rho) := \frac{1}{\Gamma(\rho)\cdot\left(\beta + x_i\right)^{\rho}} \cdot \left(y_i\right)^{\rho - 1} \cdot e^{\frac{-y_i}{\beta + x_i}}$$

We want to test the hypothesis

H_o: $\rho := 1$

H_1: ρ<>1

The following sample data has been collected

$$x := \begin{bmatrix} 12 \\ 16 \\ 18 \\ 16 \\ 12 \\ 12 \\ 16 \\ 12 \\ 10 \\ 12 \\ 16 \\ 20 \\ 12 \end{bmatrix} \qquad y := \begin{bmatrix} 20.5 \\ 31.5 \\ 47.7 \\ 26.2 \\ 44 \\ 8.28 \\ 30.8 \\ 17.2 \\ 19.9 \\ 9.96 \\ 55.8 \\ 25.2 \\ 29 \end{bmatrix}$$

$$\begin{bmatrix} 16 \\ 10 \\ 18 \\ 16 \\ 20 \\ 12 \\ 16 \end{bmatrix} \qquad \begin{bmatrix} 85.5 \\ 15.1 \\ 28.5 \\ 21.4 \\ 17.7 \\ 6.42 \\ 84.9 \end{bmatrix}$$

$ybar := mean(y)$ $\quad$ $xbar := mean(x)$

$ybar = 31.278$ $\quad$ $xbar = 14.6$

$Sy := Stdev(y)$ $\quad$ $Sx := Stdev(x)$

$Sy = 22.376$ $\quad$ $Sx = 3.119$

9.a. Likelihood Ratio Test

The unrestricted log likelihood is

$$\ln L\Omega(\beta,\rho) := \left[\left(-\rho\cdot\sum_{i=1}^{n}\ln\left(\beta+x_i\right) - n\cdot\ln(\Gamma(\rho))\right) - \left[\sum_{i=1}^{n}\left(\frac{y_i}{\beta+x_i}\right)\right]\right] + (\rho-1)\cdot\left[\sum_{i=1}^{n}\left(\ln\left(y_i\right)\right)\right]$$

The maixmum occurs where the first derivatives are zero.

$$\frac{d}{d\beta}\left[\left[\left(-\rho\cdot\sum_{i=1}^{20}\ln\left(\beta+x_i\right) - n\cdot\ln(\Gamma(\rho))\right) - \left[\sum_{i=1}^{20}\left(\frac{y_i}{\beta+x_i}\right)\right]\right] + (\rho-1)\cdot\left[\sum_{i=1}^{20}\left(\ln\left(y_i\right)\right)\right]\right]$$

$$= \quad -\rho\cdot\sum_{i=1}^{n}\frac{1}{\beta+x_i} + \left[\sum_{i=1}^{n}\left[\frac{y_i}{\left(\beta+x_i\right)^2}\right]\right]$$

$$\frac{d}{d\rho}\left[\left[\left(-\rho\cdot\sum_{i=1}^{20}\ln\left(\beta+x_i\right) - n\cdot\ln(\Gamma(\rho))\right) - \left[\sum_{i=1}^{20}\left(\frac{y_i}{\beta+x_i}\right)\right]\right] + (\rho-1)\cdot\left[\sum_{i=1}^{20}\left(\ln\left(y_i\right)\right)\right]\right]$$

$$= \quad \left(-\sum_{i=1}^{20}\ln\left(\beta+x_i\right) - n\cdot Psi(\rho)\right) + \sum_{i=1}^{20}\ln\left(y_i\right)$$

Note: Psi(ρ) is the derivative of ln(Γ(ρ)) w.r.t. ρ.

We can use mathCAD's routine's to solve the system of equations. Start by giving the routine

$\beta := -1$ $\rho := 1$

Given

$$\left(-\sum_{i=1}^{20} \ln\left(\beta + x_i\right) - n \cdot \mathrm{Psi}(\rho)\right) + \sum_{i=1}^{20} \ln\left(y_i\right) = 0$$

$$-\rho \cdot \sum_{i=1}^{20} \frac{1}{\beta + x_i} + \left[\sum_{i=1}^{20} \left[\frac{y_i}{\left(\beta + x_i\right)^2}\right]\right] = 0$$

$$\begin{bmatrix} \beta \\ \rho \end{bmatrix} := \mathrm{Find}(\beta, \rho)$$

The unrestricted estimates are

$$\begin{bmatrix} \beta \\ \rho \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The log likelihood at the unrestricted estimates is

$$L\Omega := \ln L\Omega(\beta, \rho)$$

$$L\Omega = -97.033$$

Similarly for the restricted log likelihood. The restricted log likelihood is

$$\ln L\omega(\beta) := -1 \cdot \left(\sum_{i=1}^{n} \ln\left(\beta + x_i\right)\right) - \sum_{i=1}^{n} \frac{y_i}{\beta + x_i}$$

The maximum occurs at the point where the first derivative w.r.t.β is zero.

$$\frac{d}{d\beta}\left(-\sum_{i=1}^{20}\ln\left(\beta+x_i\right)-\sum_{i=1}^{20}\frac{y_i}{\beta+x_i}\right)$$

$$=-\sum_{i=1}^{n}\frac{1}{\beta+x_i}+\sum_{i=1}^{n}\frac{y_i}{\left(\beta+x_i\right)^2}$$

We can use a MathCAD routine to find the β which makes this f.o.c. zero.

$$\mathrm{root}\left[-\sum_{i=1}^{n}\frac{1}{\beta+x_i}+\sum_{i=1}^{n}\frac{y_i}{\left(\beta+x_i\right)^2},\beta\right]=15.594$$

When ρ=1, then the constrained estimate of β is

$$\beta := 15.601$$

The value of the restricted log likelihood is

$$L\omega := \ln L\omega(\beta)$$

$$L\omega = -88.436$$

The likelihood ratio test statistic is

$$LR := -2\cdot(L\omega - L\Omega)$$

$$LR = -17.193$$

This exceeds the critical χ^2 with one degree of freedom at any reasonable level of signficance, so reject the null.

9.b. The Wald Test

To do the Wald Test we need the unrestricted estimates of the parameters

$$\begin{bmatrix}\beta\\ \rho\end{bmatrix} := \begin{bmatrix}-4.719\\ 3.151\end{bmatrix}$$

and the Information Matrix evaluated at the unrestricted estimates

$$\frac{d^2}{d\beta^2}\left[\left(-\rho\cdot\sum_{i=1}^{20}\ln\left(\beta+x_i\right)-n\cdot\ln(\Gamma(\rho))\right)-\left[\sum_{i=1}^{20}\left(\frac{y_i}{\beta+x_i}\right)\right]\right]+(\rho-1)\cdot\left[\sum_{i=1}^{20}\left(\ln\left(y_i\right)\right)\right]$$

$$=\quad \rho\cdot\sum_{i=1}^{n}\frac{1}{\left(\beta+x_i\right)^2}-2\cdot\sum_{i=1}^{n}\frac{y_i}{\left(\beta+x_i\right)^3}$$

$$\frac{d^2}{d\rho^2}\left[\left(-\rho\cdot\sum_{i=1}^{20}\ln\left(\beta+x_i\right)-n\cdot\ln(\Gamma(\rho))\right)-\left[\sum_{i=1}^{20}\left(\frac{y_i}{\beta+x_i}\right)\right]\right]+(\rho-1)\cdot\left[\sum_{i=1}^{20}\left(\ln\left(y_i\right)\right)\right]$$

$$=\quad \frac{-n\cdot\left[\left[\Gamma(\rho)\cdot\frac{d^2}{d\rho^2}\Gamma(\rho)-\left(\frac{d}{d\rho}\Gamma(\rho)\right)^2\right]\right]}{\Gamma(\rho)^2}=-7.459$$

$$\frac{d}{d\rho}\left[\frac{d}{d\beta}\left[\left[\left(-\rho\cdot\sum_{i=1}^{20}\ln\left(\beta+x_i\right)-n\cdot\ln(\Gamma(\rho))\right)-\left[\sum_{i=1}^{20}\left(\frac{y_i}{\beta+x_i}\right)\right]\right]+(\rho-1)\cdot\left[\sum_{i=1}^{20}\left(\ln\left(y_i\right)\right)\right]\right]\right]$$

$$=\quad -\sum_{i=1}^{n}\frac{1}{\beta+x_i}$$

$$I:=\begin{bmatrix}\rho\cdot\sum_{i=1}^{n}\frac{1}{\left(\beta+x_i\right)^2}-2\cdot\sum_{i=1}^{n}\frac{y_i}{\left(\beta+x_i\right)^3} & -\sum_{i=1}^{n}\frac{1}{\beta+x_i}\\ -\sum_{i=1}^{n}\frac{1}{\beta+x_i} & \frac{-n\cdot\left[\left[\Gamma(\rho)\cdot\frac{d^2}{d\rho^2}\Gamma(\rho)-\left(\frac{d}{d\rho}\Gamma(\rho)\right)^2\right]\right]}{\Gamma(\rho)^2}\end{bmatrix}$$

$$I = \begin{bmatrix} -0.856 & -2.242 \\ -2.242 & -7.459 \end{bmatrix}$$

To construct the Wald Statistic we need the variance of ρ, obtained from the information matrix as

$$Var\rho := \left(-I^{-1}\right)_{2,2}$$

$$Var\rho = 0.631$$

$$W := (\rho - 1)\cdot(Var\rho)^{-1}\cdot(\rho - 1)$$

$$W = 7.336$$

This is a large χ^2, so reject the null.

9.c. The Lagrange Multiplier Test

The restricted estimates of the unknown parameters are

$$\beta := 15.601$$

$$\rho := 1$$

The first order conditions for the maximum of the log likelihood function, evaluated at the restricted estimates are

$$-\rho\cdot\sum_{i=1}^{n}\frac{1}{\beta + x_i} + \left[\sum_{i=1}^{n}\left[\frac{y_i}{\left(\beta + x_i\right)^2}\right]\right] = 3.742\bullet 10^{-5}$$

$$\left(-\sum_{i=1}^{20}\ln\left(\beta + x_i\right) - n\cdot Psi(\rho)\right) + \sum_{i=1}^{20}\ln\left(y_i\right) =$$

The information matrix evaluated at the restricted estimates

$$I := \begin{bmatrix} \rho \cdot \sum_{i=1}^{n} \frac{1}{\left(\beta + x_i\right)^2} - 2 \cdot \sum_{i=1}^{n} \frac{y_i}{\left(\beta + x_i\right)^3} & - \sum_{i=1}^{n} \frac{1}{\beta + x_i} \\ - \sum_{i=1}^{n} \frac{1}{\beta + x_i} & \frac{-n \cdot \left[\left[\Gamma(\rho) \cdot \frac{d^2}{d\rho^2} \Gamma(\rho) - \left(\frac{d}{d\rho}\Gamma(\rho)\right)^2\right]\right]}{\Gamma(\rho)^2} \end{bmatrix}$$

$$I = \begin{bmatrix} -0.022 & -0.669 \\ -0.669 & -32.899 \end{bmatrix}$$

$$LM := \begin{bmatrix} 3.742 \cdot 10^{-5} & 7.916 \end{bmatrix} \cdot (-I)^{-1} \cdot \begin{bmatrix} 3.742 \cdot 10^{-5} \\ 7.916 \end{bmatrix}$$

$$LM = 5.116$$

Although the smallest of the three realized test statistics, this is still above even the χ^2 at the 2.5% level of test.