Hypothesis Testing Workouts: Part IV

10. The Weibull Distribution: Greene, Econometric Analysis, 3rd edition, problems 17 and 18 on pages 171-172.

The Weibull Distribution is:

$$f(x,\alpha,\beta) := \alpha\cdot\beta\cdot\left[x^{(\beta-1)}\cdot e^{-\alpha\cdot x^{\beta}}\right]$$

The corresponding likelihood function is

$$L(\alpha,\beta,x) := \prod_{i=1}^{n} \alpha\cdot\beta\cdot\left(x_i\right)^{\beta-1}\cdot e^{-\alpha\cdot\left(x_i\right)^{\beta}}$$

$$n := 20$$

$$x := \begin{bmatrix} 1.3043 \\ 1.0878 \\ .33453 \\ .49254 \\ 1.9461 \\ 1.1227 \\ 1.2742 \\ .47615 \\ 2.0296 \\ 1.4019 \\ 3.6454 \\ 1.2797 \\ .32556 \\ .15344 \\ .9608 \\ .29965 \\ 1.2357 \\ 2.007 \\ .26423 \\ .96381 \end{bmatrix}$$

10.a. Maximum likelihood estimates

The log likelihood is

$$\ln L(\alpha,\beta) := \left[n\cdot\ln(\alpha\cdot\beta) + (\beta-1)\cdot\sum_{i=1}^{n}\ln\left(x_i\right)\right] - \alpha\cdot\sum_{i=1}^{n}\left(x_i\right)^{\beta}$$

Sample data for the problem is at the right. For this data we have the following sample statistics:

$xbar := mean(x)$ $\qquad$ $s := Stdev(x)$

$xbar = 1.13$ $\qquad$ $s = 0.836$

To obtain the maximum likelihood estimates of α and β we differentiate the log likelihood w.r.t. the unknowns, set the f.o.c. to zero and solve for the estimates.

The first order conditions are

$$\frac{d}{d\alpha}\left[\left[n\cdot\ln(\alpha\cdot\beta)+(\beta-1)\cdot\sum_{i=1}^{20}\ln\left(x_i\right)\right]-\alpha\cdot\sum_{i=1}^{20}\left(x_i\right)^{\beta}\right]=0$$

$$\frac{n}{\alpha}-\sum_{i=1}^{20}\left(x_i\right)^{\beta}=0$$

$$\frac{d}{d\beta}\left[\left[n\cdot\ln(\alpha\cdot\beta)+(\beta-1)\cdot\sum_{i=1}^{20}\ln\left(x_i\right)\right]-\alpha\cdot\sum_{i=1}^{20}\left(x_i\right)^{\beta}\right]=0$$

$$\left(\frac{n}{\beta}+\sum_{i=1}^{20}\ln\left(x_i\right)\right)-\alpha\cdot\sum_{i=1}^{20}\left(x_i\right)^{\beta}\cdot\ln\left(x_i\right)=0$$

Since the f.o.c. are nonlinear in the parameters we'll use a MathCAD solve block:

Choose $\alpha := 1$ $\beta := 1$ as starting values.

Given

$$\frac{n}{\alpha}-\sum_{i=1}^{20}\left(x_i\right)^{\beta}=0$$

$$\left(\frac{n}{\beta}+\sum_{i=1}^{20}\ln\left(x_i\right)\right)-\alpha\cdot\sum_{i=1}^{20}\left(x_i\right)^{\beta}\cdot\ln\left(x_i\right)=0$$

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}:=\text{Find}(\alpha,\beta)$$

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}0.725\\ 1.444\end{bmatrix}$$

These are the unconstrained ML estimates.

At these values the log likelihood is $\ln L(\alpha,\beta)=-20.553$ $\ln L\Omega := \ln L(\alpha,\beta)$

10.b. The Wald Test: β=1

We'll need the unconstrained estimate of the variance of β from the information matrix.

Information Matrix

$$\frac{d^2}{d\alpha^2}\left[\left[n\cdot\ln(\alpha\cdot\beta)+(\beta-1)\cdot\sum_{i=1}^{20}\ln\left(x_i\right)\right]-\alpha\cdot\sum_{i=1}^{20}\left(x_i\right)^{\beta}\right]$$

$$\frac{-n}{\alpha^2}$$

$$\frac{d^2}{d\beta^2}\left[\left[n\cdot\ln(\alpha\cdot\beta)+(\beta-1)\cdot\sum_{i=1}^{20}\ln\left(x_i\right)\right]-\alpha\cdot\sum_{i=1}^{20}\left(x_i\right)^{\beta}\right]$$

$$\frac{-n}{\beta^2}-\alpha\cdot\sum_{i=1}^{20}\left(x_i\right)^{\beta}\cdot\left(\ln\left(x_i\right)\right)^2$$

$$\frac{d}{d\alpha}\frac{d}{d\beta}\left[\left[n\cdot\ln(\alpha\cdot\beta)+(\beta-1)\cdot\sum_{i=1}^{20}\ln\left(x_i\right)\right]-\alpha\cdot\sum_{i=1}^{20}\left(x_i\right)^{\beta}\right]$$

$$-\sum_{i=1}^{n}\left(x_i\right)^{\beta}\cdot\ln\left(x_i\right)$$

$$I:=\begin{bmatrix}\frac{-n}{\alpha^2} & -\sum_{i=1}^{n}\left(x_i\right)^{\beta}\cdot\ln\left(x_i\right)\\ -\sum_{i=1}^{n}\left(x_i\right)^{\beta}\cdot\ln\left(x_i\right) & \frac{-n}{\beta^2}-\alpha\cdot\sum_{i=1}^{20}\left(x_i\right)^{\beta}\cdot\left(\ln\left(x_i\right)\right)^2\end{bmatrix}$$

$$I = \begin{bmatrix} -38.082 & -14.579 \\ -14.579 & -21.818 \end{bmatrix}$$

$$Var := -I^{-1}$$

$$Var = \begin{bmatrix} 0.035 & -0.024 \\ -0.024 & 0.062 \end{bmatrix}$$

$$W := (\beta - 1)^2 \cdot \left(Var_{2,2}\right)^{-1}$$

$W = 3.198$ At the 5% level the critical χ^2 is 3.84. Do not reject the null.

10.c. The Likelihood Ratio Test: β=1

Pick the simpler of the first order conditions and set β=1, then solve for α

$$\frac{n}{\alpha} - \left[\sum_{i=1}^{20} \left(x_i\right)\right] = 0$$

$$\alpha := \frac{n}{\sum_{i=1}^{20} x_i}$$

$\alpha = 0.885$ $\beta := 1$

$$lnL\omega := \left[n \cdot \ln(\alpha \cdot \beta) + (\beta - 1) \cdot \sum_{i=1}^{n} \ln\left(x_i\right)\right] - \alpha \cdot \sum_{i=1}^{n} \left(x_i\right)^{\beta}$$

$lnL\omega = -22.449$

$$LR := -2\cdot(\ln L\omega - \ln L\Omega)$$

$LR = 3.791$ Again, do not reject the null.

10.d. The Lagrange Multiplier Test: β=1

The LM test uses the f.o.c. and -I $^{-1}$ evaluated at the restricted estimates:

$$\frac{n}{\alpha} - \sum_{i=1}^{20} (x_i)^{\beta} = 0$$

$$\left(\frac{n}{\beta} + \sum_{i=1}^{20} \ln(x_i)\right) - \alpha\cdot \sum_{i=1}^{20} (x_i)^{\beta}\cdot \ln(x_i) = 9.4$$

$$\begin{bmatrix} \dfrac{-n}{\alpha^2} & -\displaystyle\sum_{i=1}^{n} (x_i)^{\beta}\cdot \ln(x_i) \\ -\displaystyle\sum_{i=1}^{n} (x_i)^{\beta}\cdot \ln(x_i) & \dfrac{-n}{\beta^2} - \alpha\cdot \displaystyle\sum_{i=1}^{20} (x_i)^{\beta}\cdot (\ln(x_i))^2 \end{bmatrix} = \begin{bmatrix} -25.55 & -8.265 \\ -8.265 & -30.795 \end{bmatrix}$$

$$LM := (0 \quad 9.4)\cdot(-I)^{-1}\cdot\begin{bmatrix} 0 \\ 9.4 \end{bmatrix}$$

$LM = 5.442$

Well, now the conflict among the tests raises its ugly head. With this observed test statistic we would reject the null.