The Review *of* Economics *and* Statistics

VOL. LXIII AUGUST 1981 NUMBER 3

IS IT A DEMAND CURVE, OR IS IT A SUPPLY CURVE? PARTIAL IDENTIFICATION THROUGH INEQUALITY CONSTRAINTS

Edward E. Leamer*

THIS article fully describes the sets of maximum likelihood estimates of parameters in two-equation under-identified simultaneous equation systems, and uses these characterizations to comment on the usefulness of inequality constraints on the parameters. It is shown in particular that in a demand-supply system with zero covariance between the residuals and with the demand elasticity assumed to be negative and the supply elasticity assumed to be positive, the set of maximum likelihood estimates for one elasticity is the interval between the direct least-squares estimate and the reverse least-squares estimate, and the set of maximum likelihood estimates for the other parameter is the half-line in which the parameter is assumed to lie. Thus it is proper to treat the regression of quantity on price as an (attenuated) estimate of the demand curve if the estimate is negative and to treat it as an (attenuated) estimate of the supply curve if the estimate is positive. Most modern theoretical econometricians view this estimation method with incredulous amusement. In fact, the use of a method like this by Schultz (1928) can be said to have made him the reluctant mother of modern econometrics, the gang of fathers being Working (1927), Leontief (1929) and Frisch (1933). The modern exception is Maddala (1977, p. 244) who uses Frisch's (1933) observation that the probability limit of direct least squares is a weighted average of the two elasticities to conclude somewhat informally that direct least-squares is an attenuated estimator of one elasticity or the other.

Section I of this paper is an analysis of the simple supply-demand system with uncorrelated residuals. It is shown that the interval between the least-squares estimate and the reverse least-squares estimate consistently bounds one slope or the other. Knowledge of the signs of the slopes then determines whether this interval applies to the demand curve or to the supply curve. The residual variance ratio can also be bounded. If the squared correlation exceeds one-half and if quantity and price are negatively correlated, then estimates of the demand variance are necessarily less than estimates of the supply variance. This is the precise inverse form of the result of Working (1927) that the data trace out the demand curve if the supply is more variable than the demand. Of course, the opposite statement applies if the correlation is positive.

In section II, the model is generalized to admit exogenous variables, and it is shown how inequalities on the coefficients of these variables can partially identify the supply and demand slopes. The approach is similar to Marschak and Andrews (1944), but the inequalities which are considered here are weakened forms of the usual identifying restrictions. The reported algorithm for finding sets of maximum likelihood estimates subject to this form of constraint may be more useful in practice than the Marschak-Andrews treatment of non-linear constraints in production theory.

The last section contains an application of these results motivated by a paper by Houthakker (1979). Houthakker presents correlations between output and prices for fifty-nine industries, only five of which are positive, from which he concludes that prices are more influenced by supply than by demand. The application in section III is similar in approach but uses the more readily available aggregated data and also employs variables other than current price and quantity. When the equations are estimated with data in levels, four of the five industries have negative partial correlations. Only one squared

Received for publication February 19, 1980. Revision accepted for publication October 20, 1980.

* University of California, Los Angeles.

This article was stimulated by a conversation with Alan Stockman. Support from NSF Grant SOC78-09477 is gratefully acknowledged.

partial correlation is greater than one half (construction), and no conclusion about the relative variability of demand versus supply is warranted in the other four cases. When lagged variables are included in the model, all five industries have negative partial correlations between quantity and price. Only in the case of transportation is the squared partial correlation high enough to conclude that the supply curve is more variable than demand.

I. A Model without Exogenous Variables

We first consider estimates of the following simultaneous equations system:

$$Q_t = \alpha + \beta P_t + \epsilon_t \quad (1)$$
$$Q_t = \gamma + \theta P_t + u_t \qquad t = 1, \ldots, T, \quad (2)$$

where Q_t and P_t are observable, where α, β, γ and θ are fixed unobservable parameters, and where ϵ_t and u_t are serially and contemporaneously uncorrelated normal random variables with zero means and variances σ_ϵ^2 and σ_u^2. The reduced form of this model is

$$P_t = (\alpha + \epsilon_t - \gamma - u_t)/(\theta - \beta)$$
$$Q_t = ([\alpha + \epsilon_t]\theta - [\gamma + u_t]\beta)/(\theta - \beta).$$

Thus (P_t, Q_t) comes from a bivariate normal population with moments

$$E(P_t, Q_t) = (\alpha - \gamma, \alpha\theta - \gamma\beta)/(\theta - \beta) \quad (3)$$

$$V(P_t, Q_t) = \begin{bmatrix} \sigma_\epsilon^2 + \sigma_u^2 & \theta\sigma_\epsilon^2 + \beta\sigma_u^2 \\ \theta\sigma_\epsilon^2 + \beta\sigma_u^2 & \theta^2\sigma_\epsilon^2 + \beta^2\sigma_u^2 \end{bmatrix} (\theta - \beta)^{-2}. \quad (4)$$

Maximum likelihood estimation in this case requires that the sample moments be set equal to the population moments (3) and (4). If θ and β were known, the sample means could be used to solve uniquely for α and γ using equations (3), provided that $\beta \neq \theta$. Since otherwise this places no restrictions on β and θ, it is the sample variance matrix which must be relied on to determine β and θ. Setting sample moments equal to population moments yields the three equations

$$(\hat\theta - \hat\beta)^2 s_p^2 = \hat\sigma_\epsilon^2 + \hat\sigma_u^2$$
$$(\hat\theta - \hat\beta)^2 s_q^2 = \hat\theta^2\hat\sigma_\epsilon^2 + \hat\beta^2\hat\sigma_u^2$$
$$(\hat\theta - \hat\beta)^2 s_{pq} = \hat\theta\hat\sigma_\epsilon^2 + \hat\beta\hat\sigma_u^2$$

where s_p^2 and s_q^2 are sample variances and s_{pq} is the sample covariance. The first and last of these equations can be written as

$$\begin{bmatrix} \hat\sigma_\epsilon^2 \\ \hat\sigma_u \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \hat\theta & \hat\beta \end{bmatrix}^{-1} \begin{bmatrix} s_p^2 \\ s_{pq} \end{bmatrix} (\hat\theta - \hat\beta)^2$$
$$= \begin{bmatrix} \hat\beta s_p^2 - s_{pq} \\ -\hat\theta s_p^2 + s_{pq} \end{bmatrix} (\hat\beta - \hat\theta)^{-1}(\hat\theta - \hat\beta)^2 \quad (5)$$

which can be inserted into the second to produce

$$\hat\theta^2(\hat\beta s_p^2 - s_{pq}) + \hat\beta^2(-\hat\theta s_p^2 + s_{pq}) = (\hat\beta - \hat\theta)s_q^2,$$

which in turn can be rewritten as

$$(\hat\theta - b)(\hat\beta - b) = (r_{pq}^2 - 1)s_q^2/s_p^2 \quad (6)$$

where b is the ordinary least-squares estimate

$$b = s_{pq}/s_p^2,$$

and r_{pq}^2 is the squared sample correlation

$$r_{pq}^2 = s_{pq}^2/s_p^2 s_q^2.$$

The set of maximum likelihood estimates of β and θ therefore is the hyperbola (6) intersected with the region which determines positive estimates of the variances (5), that is,

$$(\hat\beta - b)/(\hat\beta - \hat\theta) \geq 0$$
$$(\hat b - \theta)/(\hat\beta - \hat\theta) \geq 0,$$

but these constraints are redundant since they are implied by (6) using the information that $r_{pq}^2 \leq 1$.

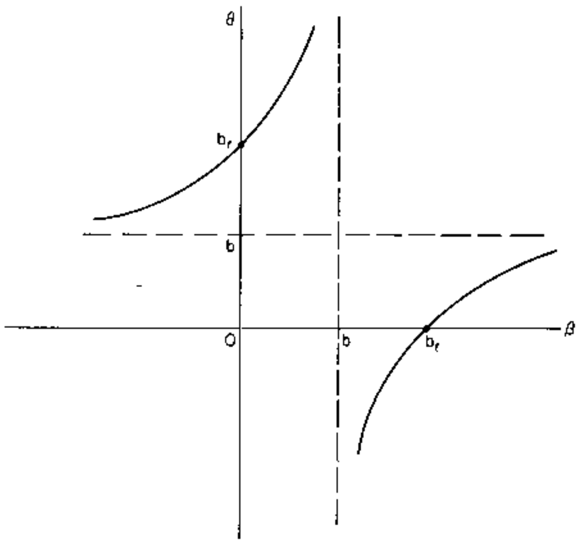
The hyperbola of maximum likelihood estimates (6) is graphed in figure 1 given the assumption that the least-squares estimate is positive. It may be noted that any estimate of β is possible, and any estimate of θ; but given one, there is a unique maximum likelihood estimate of the other. In particular, if one is known to be zero, then the estimate of the other is the reverse regression estimate

$$b_r = s_q^2/s_{pq} = b/r_{pq}^2; \quad (7)$$

or if one is known to be infinite, then the estimate of the other is least-squares, b.

The usefulness of inequality constraints is clear from figure 1. If equation (1) is taken to be the demand curve, $\beta < 0$, then only the part of the hyperbola to the left of the θ-axis is relevant. As drawn, the maximum likelihood estimates of the supply slope θ must lie between least-squares and reverse least-squares: $0 < b < \hat\theta < b_r$. Knowledge that the supply slope is positive, $\theta > 0$, restricts the estimate of β to be either

FIGURE 1.—MAXIMUM LIKELIHOOD ESTIMATES OF β AND θ
$b = s_{pq}/s_p^2,\ b_r = s_q^2/s_{pq} = b/r_{pq}^2$



larger than reverse least-squares or smaller than least-squares: $\hat{\beta} < b$ or $b_r < \hat{\beta}$. Together, the inequalities $\beta < 0$, $\theta > 0$ and a positive least-squares estimate, $b > 0$, imply the inequalities

$$\hat{\beta} < 0, \quad 0 < b < \hat{\theta} < b_r. \tag{8}$$

Alternatively, if the least-squares estimate is negative, these inequalities become

$$b_r < \hat{\beta} < b < 0, \quad 0 < \hat{\theta}. \tag{9}$$

In other words, when the regression of quantity on price yields a positive estimate, we may assume that this is an attenuated estimate of the supply curve and that the data contain no useful information about the demand curve. If the estimate is negative, the number may be treated as an attenuated estimate of the demand slope, and we may assert that the data contain no useful information about the supply curve.

Under general conditions, maximum likelihood estimators are consistent, and there is nothing in this problem to suggest that (8) and (9) are not consistent bounds. It is easy to show this, using the fact that b and b_r are consistent estimates of the corresponding population moments

$$\text{plim}\ (b) = (\theta\sigma_\epsilon^2 + \beta\sigma_u^2)/(\sigma_\epsilon^2 + \sigma_u^2) \tag{10}$$
$$\text{plim}\ (b_r) = (\theta^2\sigma_\epsilon^2 + \beta^2\sigma_u^2)/(\theta\sigma_\epsilon^2 + \beta\sigma_u^2). \tag{11}$$

Equation (10) is a weighted average of θ and β. Therefore as shown by Maddala (1977, p. 244), $\beta \leq \text{plim}\ (b) \leq \theta$. The other bound can be demonstrated by exploiting the symmetry of the problem. From $\text{plim}\ (b_r^{-1}) = (\theta^{-1}\theta^2\sigma_\epsilon^2 + \beta^{-1}\beta^2\sigma_u^2)/(\theta^2\sigma_\epsilon^2 + \beta^2\sigma_u^2)$, we obtain $\beta^{-1} \leq \text{plim}\ (b_r^{-1}) \leq \theta^{-1}$. Thus if $\text{plim}\ (b_r^{-1}) < 0$, then $\text{plim}\ (b_r) < \beta$.

Bounds for the coefficient estimates imply bounds for estimates of the variance ratio which is the following function of $\hat{\beta}$ and $\hat{\theta}$:

$$\begin{aligned}\hat{\sigma}_\epsilon^2/\hat{\sigma}_u^2 &= (\hat{\beta}s_p^2 - s_{pq})/(s_{pq} - \hat{\theta}s_p^2)\\ &= (\hat{\beta} - b)/(b - \hat{\theta}).\end{aligned}$$

In the case when least-squares is positive $b > 0$, as the estimates of (β, θ) vary from $(0, b_r)$ to $(-\infty, b)$ the variance ratio varies from $-s_{pq}/(s_{pq} - s_q^2 s_p^2/s_{pq}) = 1/(r_{pq}^{-2} - 1)$ to ∞:

$$(r_{pq}^{-2} - 1)^{-1} < \hat{\sigma}_\epsilon^2/\hat{\sigma}_u^2 < \infty.$$

If r_{pq}^2 exceeds one-half, the lower bound exceeds one, and estimates of the demand variance necessarily exceed estimates of the supply variance; otherwise one is a possible estimate for the variance ratio. Thus it is necessary but not sufficient to have a negative correlation in order to conclude that the supply curve is more unstable than the demand. Similarly, if $b < 0$, the variance ratio is bounded by

$$0 < \hat{\sigma}_\epsilon^2/\hat{\sigma}_u^2 < r_{pq}^{-2} - 1,$$

with $r_{pq}^2 > 1/2$ again necessary to bound the estimate away from one.

It is of historical interest to note that Leontief (1929) includes the hyperbola (6) although expressed in the form $\hat{\beta} = (\hat{\theta}s_{pq} - s_q^2)/(\hat{\theta}s_p^2 - s_{pq})$. Actually he uses a pair of hyperbolas formed by splitting the data set into two parts, and is able to solve for a pair of estimates which jointly satisfy both hyperbolic equalities. This procedure brought down upon him the wrath of Frisch's (1933) *Pitfalls*, which is devoted almost completely to debunking the method.

There is of course a problem with the Leontief procedure. If the variances σ_u^2 and σ_ϵ^2 are the same in both halves of the sample, then asymptotically the two least-squares estimates will necessarily coincide, as will the two hyperbolas. Moreover, the two estimates of the variance ratio necessarily differ in any finite sample. The method thus rests on the unlikely assumption that the slopes β and θ are constant over time but the variances are not. Still, Leontief did have the hyperbola properly defined, which is only one short step from the results in this paper. It is therefore surprising that Leontief's contribution has been so completely ignored by the post-1940 econometrics literature. The fault seems to me to lie with excessive attention to asymptotic properties of estimators and insufficient interest in the shapes of likelihood functions.

II. A Model with Exogenous Variables

We next consider a more general model in which an observable variable x_t affects both the quantity supplied and the quantity demanded. If there is no a priori information about the signs of the new coefficients, then the set of maximum likelihood estimates of β and θ is altered only in that the sample moments are computed after controlling for x_t. In other words, the direct and reverse regressions include the variable x_t. If there are equality or inequality constraints on the x-coefficients, then the set of estimates of β and θ may change more dramatically.

The model (1) and (2) is altered to allow α and γ to be functions of x:

$$\alpha = \alpha_0 + \alpha_1 x_t$$
$$\gamma = \gamma_0 + \gamma_1 x_t.$$

This leaves the variance (4) unchanged but alters the means to

$$E(P_t, Q_t|x_t) = (\alpha_0 - \gamma_0 + [\alpha_1 - \gamma_1]x_t, \alpha_0\theta - \gamma_0\beta + [\alpha_1\theta - \gamma_1\beta]x_t)/(\theta - \beta).$$

The two x-coefficients in this reduced form are estimated by regressing P_t on x_t and Q_t on x_t, respectively. These will be indicated by d_p and d_q

$$\begin{bmatrix} s_{px}/s_x^2 \\ s_{qx}/s_x^2 \end{bmatrix} \equiv \begin{bmatrix} d_p \\ d_q \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_1 - \hat{\gamma}_1 \\ \hat{\alpha}_1\hat{\theta} - \hat{\gamma}_1\hat{\beta} \end{bmatrix} (\hat{\theta} - \hat{\beta})^{-1},$$

which can be solved for $(\hat{\alpha}_1, \hat{\gamma}_1)$ in terms of $(\hat{\theta}, \hat{\beta})$

$$\begin{bmatrix} \hat{\alpha}_1 \\ \hat{\gamma}_1 \end{bmatrix} = \begin{bmatrix} d_q - \hat{\beta} d_p \\ d_q - \hat{\theta} d_p \end{bmatrix}. \quad (12)$$

The logic that leads to estimates of β and θ is the same as before except that the sample moments control for x. The least-squares estimate of b in (6) is therefore replaced by least-squares controlling for x, which for ease of notation will still be denoted by b:

$$b = b_{pq \cdot x} = s_{pq \cdot x}/s_{p \cdot x}^2..$$

The other moments are similarly altered. In particular, the reverse regression estimate, formed first by regressing p on q and x and then solving the equation for q as a function of p and x, is

$$b_r = b/r_{pq \cdot x}^2$$

where

$$r_{pq \cdot x}^2 = s_{pq \cdot x}^2/s_{p \cdot x}^2 s_{q \cdot x}^2.$$

Thus the couple $(\hat{\beta},\hat{\theta})$ lies on a hyperbola centered at least-squares, and the x-coefficients are computed using (12).

For the sake of interpretation, it may be shown using equation (12) that if $\hat{\beta}$ is set equal to the least-squares value, then $\hat{\alpha}_1$ is also equal to the least-squares value, and if $\hat{\beta}$ is reverse least-squares than $\hat{\alpha}_1$ is reverse least-squares. The least-squares estimate of α_1 given β is formed by regressing $Q_t - \beta P_t$ on x_t

$$\hat{\alpha}_1 = s_x^{-2}[s_{xq} - \beta s_{xp}] = d_q - \beta d_p,$$

which is just equation (12). The reverse regression estimate of $\hat{\alpha}_1$ is computed by first estimating the reverse equation $P_t = (Q_t - \alpha_0 - \alpha_1 x_t - \epsilon_t)/\beta$ and then letting $\hat{\alpha}_1$ be the coefficient on x divided by the coefficient on Q. Given β, this estimate is formed by regressing $P_t - \beta^{-1}Q_t$ on x_t:

$$\hat{\alpha}_1 = -\beta\, s_x^{-2}\, [s_{xp} - \beta^{-1} s_{xq}] = d_q - \beta d_p,$$

which is just equation (12).

Now we consider the usefulness of information about α_1 and γ_1 for estimating β and θ. The traditional identifying restriction is that the coefficient of the exogenous variable is zero in one equation. If α_1 is known to be zero, then (12) determines the estimate of β

$$\hat{\beta} = d_q/d_p \equiv b_{IV},$$

which is the instrumental variables estimate or equivalently the two-stage least-squares estimate. The other equation is also identified since given $\hat{\beta}$ we can solve for $\hat{\theta}$ using hyperbola (6)

$$\hat{\theta} = b_H = b_{pq \cdot x} + (r_{pq \cdot x}^2 - 1)s_{q \cdot x}^2/ s_{p \cdot x}^2(b_{IV} - b_{pq \cdot x}),$$

which will henceforth be called the "hyperbolic estimate."

Less precise knowledge of α_1 may also be useful. The exact algorithm for finding maximum likelihood estimates given inequality constraints on α_1 and γ_1 as well as β and θ is tedious, but not difficult. Given equations (12), the sign pattern for α_1 and γ_1 selects a quadrant located at (b_{IV},

b_{IV}) within which ($\hat{\beta}$, $\hat{\theta}$) must lie. This may or may not further restrict $\hat{\beta}$ and $\hat{\theta}$ given that they are already restricted to lie in the quadrant $\hat{\beta} < 0$, $\hat{\theta} > 0$. Suppose, for example, that it is known that $\alpha_1 > 0$ and $\gamma_1 > 0$. Then from (12), estimates must be chosen such that $d_q - \hat{\beta} d_p > 0$ and $d_q - \hat{\theta} d_p > 0$. These inequalities sometimes further restrict the set of estimates. A listing of all cases is provided in table 1, where "no solution" means that the constraints are mutually contradictory in the sense that the sample moments violate the assumed constraints. If this is treated as a small sample aberration, then maximum likelihood estimation can be done given the list of constraints. This will ordinarily imply a unique maximum likelihood estimate.

The general case with many exogenous variables is also straightforward, though the bookkeeping is still more involved. The constants α and γ in (1) and (2) become

$$\alpha = \alpha_0 + \sum_{i=1}^{I} \alpha_i x_{ti}$$

$$\gamma = \gamma_0 + \sum_{i=1}^{I} \gamma_i x_{ti}$$

and equation (12) becomes

$$\begin{bmatrix} \hat{\alpha}_i \\ \hat{\gamma}_i \end{bmatrix} = \begin{bmatrix} d_{qi} - \hat{\beta} d_{pi} \\ d_{qi} - \hat{\theta} d_{pi} \end{bmatrix}, \; i = 1, \ldots, I \qquad (13)$$

where d_q and d_p are the vectors of reduced form coefficients from the two equations. If it is known that $\alpha_i = 0$, then equation (13) implies the consistent estimate of β,

$$\hat{\beta} = b_{IV}(i) \equiv d_{qi}/d_{pi},$$

which is the two-stage least-squares estimator, or, equivalently, the instrumental variables estimator with x_{ti} serving as an instrument for P_t.

Estimation with inequality constraints on the coefficients proceeds as follows. Let the given restrictions be $\alpha_i \operatorname{sgn}(\alpha_i) \geq 0$ and $\gamma_i \operatorname{sgn}(\gamma_i) \geq 0$ where sgn is the sign function with sgn $= 0$ if the restriction is not assumed. Then these inequalities generate restrictions on $\hat{\beta}$ and $\hat{\theta}$ from equation (13)

$$\operatorname{sgn}(\alpha_i)\,(d_{qi} - \hat{\beta} d_{pi}) \geq 0$$
$$\operatorname{sgn}(\gamma_i)\,(d_{qi} - \hat{\theta} d_{pi}) \geq 0.$$

If these restrictions, together with $\beta \operatorname{sgn}(\beta) > 0$ and $\theta \operatorname{sgn}(\theta) > 0$, are not mutually exclusive they identify a box of feasible estimates $\hat{\beta}_L < \hat{\beta} < \hat{\beta}_M$, $\hat{\theta}_L < \hat{\theta} < \hat{\theta}_M$. This box is intersected with the hyperbola of estimates to form the set of maximum likelihood estimates. If it is known that $\beta > 0$ and $\theta < 0$, then the box lies in the second quadrant, and when intersected with the hyperbola produces a curve segment of estimates. At the end points of this hyperbolic segment, either $\hat{\beta}$ or $\hat{\theta}$ will equal an instrumental variables estimate with one of the exogenous variables serving as an instrument for P. In that sense, the above describes an algorithm for selecting an instrumental variable from a list of candidates.[1]

[1] Inequality constraints in a multi-equation model are more difficult to use. The model is written as $BY = \Gamma z + u$, where Y is the vector of endogenous variables, z is the vector of exogenous variables and u is the vector of residuals, normally distributed with zero means and diagonal covariance matrix D. Regressing Y on z produces the reduced form coefficients π and the residual covariance matrix S. Given B, we may solve for estimates of Γ and D: $\Gamma = B\pi$ and $D = BSB'$. Inequalities for Γ imply linear inequalities on estimates of B, but the diagonality of D implies that pairs of rows of B, say β_i and β_j, have zero inner products $\beta'_i S \beta_j = 0$. This sequence of quadratic constraints is difficult to work with except in the two-dimensional case.

TABLE 1.—CONSTRAINTS ON MAXIMUM LIKELIHOOD ESTIMATES
$\beta < 0$, $\theta > 0$, $\alpha_1 > 0$, $\gamma_1 > 0$

Estimates				Constraints on $\hat{\beta}$		Constraints on $\hat{\theta}$		Constraints on $\hat{\sigma}_\epsilon^2/\hat{\sigma}_u^2$	
b	d_p	b_{IV}		Lower	Upper	Lower	Upper	Lower	Upper
+	+	+	$b_r < b_{IV}$	$-\infty$	0	b	b_r	$(r^{-2} - 1)^{-1}$	∞
+	+	+	$b < b_{IV} < b_r$	$-\infty$	b_H	b	b_{IV}	V	∞
+	+	±	$b_{IV} < b$	.	No solution	.	.	.	.
+	−	+		.	No solution	.	.	.	.
+	−	−		b_{IV}	0	b_H	b_r	$(r^{-2} - 1)^{-1}$	V^{-1}
−	+	+		b_r	b_H	0	b_{IV}	V	$r^{-2} - 1$
−	+	−		.	No solution	.	.	.	.
−	−	±	$b < b_{IV}$	.	No solution	.	.	.	.
−	−	−	$b_r < b_{IV} < b$	b_{IV}	b	b_H	∞	0	V^{-1}
−	−	−	$b_{IV} < b$	b_r	b	0	∞	0	$r^{-2} - 1$

$r^2 = s_{pq\cdot z}^2/s_{p\cdot z}^2 s_{q\cdot z}^2$, $V = (b_H - b)/(b - b_{IV})$

III. An Example

This section contains an example based on U.S. annual data for five aggregated industries. The quantity variable is the industry's product in 1972 dollars. The price variable is the industry's implicit deflator divided by the GNP deflator. Current and constant dollar product, and the GNP deflator 1947–1978 are all taken from *The National Income and Product Accounts of the United States*, 1929–1974, *Statistical Tables*, supplement to the *Survey of Current Business*, and from the July 1977 and 1979 volumes of the *Survey*. The only other variable is the total civilian employment with data from *Business Conditions Digest*, March 1980, and for 1947 from *Historical Statistics of the U.S.*

A complete econometric study of the demand and supply relationships in these aggregated industries would necessarily involve a more complicated model. For the purpose of illustrating the usefulness of inequality constraints, the simple model is adequate. For each industry, I am supposing that all other goods can be aggregated into the composite "GNP." The size of the civilian employment determines the size of the production possibilities frontier relating the maximum possible output of the industry to different levels of output of the alternative product "GNP." The supply curve of the industry associates with any given slope of the production frontier (i.e., relative price) a level of the industry's output. Demand is assumed to result from the maximization of a well behaved utility function subject to a consumption possibilities constraint, which depends on the relative price and real income, and therefore on the relative price and the labor force. Thus the labor force is assumed to have a positive effect on both quantity demanded and quantity supplied.

This formulation obviously leaves out many complications. It includes no measure of capital. Labor is assumed to be costlessly mobile between industries. Expectations play no role. The aggregation issues are entirely ignored. Be that as it may, the regressions of quantity on price and labor force, and price on quantity and labor force are reported in table 2. The latter regression is inverted to form the reverse regression. For four of the five industries, the estimated price elasticity is negative, and if there were no knowledge of the sign of the labor force variables, we would take this to be an attenuated estimate of the demand elasticity. The intervals between the direct and reverse regressions are sometimes short enough to be useful. Agricultural demand is estimated to be rather inelastic, as is mining. Transportation and construction have relatively high estimated elasticities.

In order to make use of the inequalities on the coefficients of the employment variable, it is necessary to compute the reduced form, reported in table 3. The relevant parts of tables 2 and 3 are inserted into table 4, and bounds are computed by referring to table 1. In the case of the agricultural industry, the additional information did not prove useful, and estimates of the demand elasticity are constrained to the interval between the

TABLE 2.—ESTIMATED REGRESSIONS, STRUCTURAL FORM

Sector	Quantity	Price	Employment	R^2	D.W.	$r_{pq\cdot x}^2$
Agriculture	1	−0.11 (.03)	0.55 (.04)	.90	1.27	.26
	−2.4 (.74)	1	1.00 (.47)	.35	.63	
(Reverse Reg.)	1	−0.42	0.42			
Construction	1	−1.67 (.19)	2.72 (.16)	.92	.45	.73
	−0.44 (.05)	1	1.38 (.09)	.89	.43	
(Reverse Reg.)	1	−2.27	3.13			
Manufacturing	1	0.80 (.43)	2.55 (.23)	.95	.42	.10
	0.13 (.07)	1	−0.78 (.16)	.86	.29	
(Reverse Reg.)	1	7.61	5.92			
Mining	1	−0.19 (.04)	1.22 (.05)	.96	.93	.41
	−2.1 (.48)	1	2.55 (.62)	.41	.41	
(Reverse Reg.)	1	−0.47	1.19			
Transportation	1	−0.78 (.15)	1.33 (.07)	.98	.74	.48
	−0.60 (.12)	1	0.61 (.19)	.82	.92	
(Reverse Reg.)	1	−1.66	1.01			

Note: All variables in logarithms; standard errors in parentheses; D.W. = Durbin-Watson statistic.

TABLE 3.—ESTIMATED REGRESSIONS, REDUCED FORM

	Employment	R^2	D.W.	Residual Sum-of-Squares	
				Quantity	Price
Agriculture					
Quantity	0.60 (.04)	.86	1.12	.039	−.091
Price	−0.41 (.20)	.12	0.43	−.091	.823
Construction					
Quantity	1.60 (.19)	.71	0.11	.732	−.322
Price	0.67 (.10)	.62	0.14	−.322	.193
Manufacturing					
Quantity	2.15 (.09)	.95	0.34	.184	.024
Price	−0.49 (.04)	.84	0.19	.024	.030
Mining					
Quantity	1.24 (.06)	.93	0.50	.080	−.172
Price	−0.10 (.21)	.01	0.20	−.172	.911
Transportation					
Quantity	1.60 (.05)	.97	0.96	.057	−.035
Price	−0.36 (.05)	.66	1.27	−.035	.045

Note: All variables in logarithms; standard errors in parentheses; D.W. = Durbin-Watson statistic.

reverse and direct regressions. No constraint applies to $\hat{\theta}$, and the partial R^2 is not high enough to assure that the estimates of the demand residual variance are necessarily less than estimates of the supply residual variance.

In contrast, the extra information is useful for estimating the construction elasticities. The labor force variable can be used as an instrument for price when estimating the supply elasticity, in the sense that the upper bound for $\hat{\theta}$ is the instrumental variables estimate, 2.38. The corresponding "hyperbolic" estimate of β, −1.92, becomes the upper bound for $\hat{\beta}$. Notice that construction is the only industry in which estimates of the demand variance are necessarily less than estimates of the supply variance.

For manufacturing, the inequalities on the labor force coefficients also prove useful, and the labor force variable becomes an instrumental variable for estimating the demand curve. For mining and transportation, the extra inequalities prove useless and only the demand elasticity is constrained.

The result that estimates of the construction supply variance necessarily exceed estimates of the demand variance is troubling since one might have supposed at the outset that construction demand is highly variable. Measurement error, especially in the price variable, can always be used as an explanation. The dependent variable "quantity" is actually a measure of value of output divided by the price index, and as a result will have a negative correlation with price if measurement errors are important enough. In his study of more disaggregated data Houthakker (1979, p. 249) writes, "Although an entirely conclusive answer cannot be given, my own tentative judgment is that measurement error is not the main source of the negative correlation." I am somewhat more suspicious.

Another source of concern with these estimates is that the model incorporates no time-series phenomena, yet the low Durbin-Watson statistics in table 2 point to the importance of lagged variables. For this reason, I will now include in the model lagged values of employment,

TABLE 4.—ESTIMATES AND BOUNDS FOR PRICE ELASTICITIES AND RESIDUAL VARIANCE RATIO[a]

	Estimates					Bound for $\hat{\beta}$		Bound for $\hat{\theta}$		Bound for $\hat{\sigma}_\epsilon^2/\hat{\sigma}_u^2$	
	b	b_r	d_p	d_q	b_{IV}	Lower	Upper	Lower	Upper	Lower	Upper
Agriculture	−0.11	−0.42	−0.41	0.60	−1.46	−0.42	−0.11	0	∞	0	2.8
Construction	−1.67	−2.27	0.67	1.60	2.38	−2.27	−1.92	0	2.38	0.06	0.4
Manufacturing	0.80	7.61	−0.49	2.15	−4.39	−4.39	0	1.86	7.61	0.11	4.9
Mining	−0.19	−0.47	−0.10	1.24	−12.4	−0.47	−0.19	0	∞	0	1.4
Transportation	−0.78	−1.66	−0.36	1.60	−4.44	−1.66	−0.78	0	∞	0	1.1

[a] Based on estimates from table 3.

TABLE 5.—ESTIMATED REGRESSIONS, REDUCED FORM

	Emp	Emp_{-1}	P_{-1}	Q_{-1}	R^2	Residual Sum-of-Squares Quantity	Price
Agriculture							
Quantity	−0.189	0.512	−.027	.439	.92	.018	−.0063
Price	2.002	−1.836	.769	−.313	.72	−.0063	.215
Construction							
Quantity	2.071	−1.886	−.193	.822	.99	.0230	−.0144
Price	0.249	−0.020	.849	−.041	.95	−.0144	.0227
Manufacturing							
Quantity	3.45	−3.05	.265	.806	.99	.0324	−.0066
Price	−0.37	0.358	.877	−.031	.98	−.0066	.0037
Mining							
Quantity	2.255	−2.081	−.024	.781	.98	.0254	−.0144
Price	−1.025	2.344	.802	−.932	.86	−.0144	.1235
Transportation							
Quantity	3.024	−2.75	.533	.924	.99	.0172	−.0129
Price	−0.519	0.543	.259	−.197	.87	−.0129	.0164

price and quantity. The coefficients of lagged quantity as well as current employment will be assumed to be positive in both the supply and the demand equations. The other coefficients will not be constrained because their signs seem ambiguous. If the lagged variables are in the model entirely because the residuals are first-order autocorrelated, then the lagged variables will have signs opposite of the current variables. But if a distributed lag model is hypothesized, the lagged variables are likely to have the same sign as the current variable. Incidentally, the neglect of expectations may be more serious in the model with lagged influences.

The reduced form estimates are reported in table 5, and the implied bounds are reported in table 6. Because only the employment and the lagged quantity coefficients are constrained, only these two variables can be used as instruments. As it turns out, current employment is selected as an instrument for estimating the construction supply elasticity and lagged output is selected as an instrument for estimating the mining demand elasticity. Agriculture admits no solution because the employment variable is estimated in the reduced form to have a negative effect on quantity and a positive effect on price, which violates the assumptions of the model.

The differences between tables 4 and 6 are not too great. In both cases the information is better about the demand elasticities than the supply elasticities. The ordering of the demand elasticities is similar. In table 6, the estimates of the ratio of the demand variance to the supply variance for construction are not less than one, but they are less than one for transportation.

IV. Conclusion

This article has a specific and a general conclusion. The specific conclusion is that, contrary to prevailing opinion, it does make sense to regress quantity on price and then to take the estimated function to be a supply curve or a demand curve depending on the sign of the estimated elasticity. To do this it is necessary to assume that the

TABLE 6.—ESTIMATES AND BOUNDS FOR PRICE ELASTICITIES AND RESIDUAL VARIANCE[a]

	Estimates				Bound for $\hat{\beta}$		Bound for $\hat{\theta}$		Bound for $\hat{\sigma}_\epsilon^2/\hat{\sigma}_u^2$	
	b	b_r	$b_{IV}(EMP)$	$b_{IV}(Q_{-1})$	Lower	Upper	Lower	Upper	Lower	Upper
Agriculture	−0.029	−2.8	−0.094	−1.40	No solution		No solution		No solution	
Construction	−0.63	−1.6	8.32	−20.0	−1.6	−0.70	0	8.32	0.008	1.56
Manufacturing	−1.81	−4.9	−9.32	−26.0	−4.9	−1.81	0	∞	0	1.75
Mining	−0.12	−1.7	−2.20	−0.84	−0.84	−0.12	0.15	∞	0	3.86
Transportation	−0.79	−1.3	−5.82	−4.69	−1.3	−0.79	0	∞	0	0.69

[a] Based on estimates from table 5.

covariance between the errors in the supply and demand equations is zero. Although the estimated slope is biased downward, the reverse regression estimate consistently overestimates the true slope, and consequently the true slope can be consistently bounded.

The general conclusion of this article is that zero covariance restrictions together with inequality constraints on parameters can serve to partially identify under-identified systems. In the two-equation case, algorithms for imposing these constraints in effect select an instrumental variable from a candidate list.

REFERENCES

Frisch, Ragnar, *Pitfalls in the Statistical Construction of Demand and Supply Curves* (Leipzig: Hans Buske, Verlag, 1933).

Houthakker, Hendrik S., "Growth and Inflation: Analysis by Industry," *Brookings Papers on Economic Activity* (1: 1979), 241–257.

Leontief, Wassily, "Ein Versuch Zur Statistichen Analyse von Angebot und Nachfrage," *Weltwirtschaftliches Archiv*, 30, Heft 1 (1929), 1–53.

Maddala, G. S., *Econometrics* (New York: McGraw-Hill, 1977).

Marschak, Jacob, and William H. Andrews, "Random Simultaneous Equations and the Theory of Production," *Econometrica* 12 (3–5) (1944), 143–205.

Schultz, Henry, *Statistical Laws of Demand and Supply* (Chicago: The University of Chicago Press, 1928).

Working, Elmer, "What Do Statistical 'Demand Curves' Show?," *Quarterly Journal of Economics* 41 (1927), 212–235.