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This paper uses an intertemporal general equilibrium asset pricing model to study the term structure of interest rates. In this model, anticipations, risk aversion, investment alternatives, and preferences about the timing of consumption all play a role in determining bond prices. Many of the factors traditionally mentioned as influencing the term structure are thus included in a way which is fully consistent with maximizing behavior and rational expectations. The model leads to specific formulas for bond prices which are well suited for empirical testing.

1. INTRODUCTION

THE TERM STRUCTURE of interest rates measures the relationship among the yields on default-free securities that differ only in their term to maturity. The determinants of this relationship have long been a topic of concern for economists. By offering a complete schedule of interest rates across time, the term structure embodies the market's anticipations of future events. An explanation of the term structure gives us a way to extract this information and to predict how changes in the underlying variables will affect the yield curve.

In a world of certainty, equilibrium forward rates must coincide with future spot rates, but when uncertainty about future rates is introduced the analysis becomes much more complex. By and large, previous theories of the term structure have taken the certainty model as their starting point and have proceeded by examining stochastic generalizations of the certainty equilibrium relationships. The literature in the area is voluminous, and a comprehensive survey would warrant a paper in itself. It is common, however, to identify much of the previous work in the area as belonging to one of four strands of thought.

First, there are various versions of the expectations hypothesis. These place predominant emphasis on the expected values of future spot rates or holding-period returns. In its simplest form, the expectations hypothesis postulates that bonds are priced so that the implied forward rates are equal to the expected spot rates. Generally, this approach is characterized by the following propositions: (a) the return on holding a long-term bond to maturity is equal to the expected return on repeated investment in a series of the short-term bonds, or (b) the expected rate of return over the next holding period is the same for bonds of all maturities.

The liquidity preference hypothesis, advanced by Hicks [**16**], concurs with the importance of expected future spot rates, but places more weight on the effects of the risk preferences of market participants. It asserts that risk aversion will cause forward rates to be systematically greater than expected spot rates, usually

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by an amount increasing with maturity. This term premium is the increment required to induce investors to hold longer-term ("riskier") securities.

Third, there is the market segmentation hypothesis of Culbertson [7] and others, which offers a different explanation of term premiums. Here it is asserted that individuals have strong maturity preferences and that bonds of different maturities trade in separate and distinct markets. The demand and supply of bonds of a particular maturity are supposedly little affected by the prices of bonds of neighboring maturities. Of course, there is now no reason for the term premiums to be positive or to be increasing functions of maturity. Without attempting a detailed critique of this position, it is clear that there is a limit to how far one can go in maintaining that bonds of close maturities will not be close substitutes. The possibility of substitution is an important part of the theory which we develop.

In their preferred habitat theory, Modigliani and Sutch [25] use some arguments similar to those of the market segmentation theory. However, they recognize its limitations and combine it with aspects of the other theories. They intended their approach as a plausible rationale for term premiums which does not restrict them in sign or monotonicity, rather than as a necessary causal explanation.[2]

While the focus of such modern and eclectic analyses of the term structure on explaining and testing the term premiums is desirable, there are two difficulties with this approach. First, we need a better understanding of the determinants of the term premiums. The previous theories are basically only hypotheses which say little more than that forward rates should or need not equal expected spot rates. Second, all of the theories are couched in ex ante terms and they must be linked with ex post realizations to be testable.

The attempts to deal with these two elements constitute the fourth strand of work on the term structure. Roll [29, 30], for example, has built and tested a mean-variance model which treated bonds symmetrically with other assets and used a condition of market efficiency to relate ex ante and ex post concepts.[3] If rationality requires that ex post realizations not differ systematically from ex ante views, then statistical tests can be made on ex ante propositions by using ex post data.

We consider the problem of determining the term structure as being a problem in general equilibrium theory, and our approach contains elements of all of the previous theories. Anticipations of future events are important, as are risk preferences and the characteristics of other investment alternatives. Also, individuals can have specific preferences about the timing of their consumption, and thus have, in that sense, a preferred habitat. Our model thus permits detailed predictions about how changes in a wide range of underlying variables will affect the term structure.

[2] We thank Franco Modigliani for mentioning this point.

[3] Stiglitz [35] emphasizes the portfolio theory aspects involved with bonds of different maturities, as do Dieffenbach [9], Long [18], and Rubinstein [31], who incorporate the characteristics of other assets as well. Modigliani and Shiller [24] and Sargent [33] have stressed the importance of rational anticipations.

The plan of our paper is as follows. Section 2 summarizes the equilibrium model developed in Cox, Ingersoll, and Ross [**6**] and specializes it for studying the term structure. In Section 3, we derive and analyze a model which leads to a single factor description of the term structure. Section 4 shows how this model can be applied to other related securities such as options on bonds. In Section 5, we compare our general equilibrium approach with an alternative approach based purely on arbitrage. In Section 6, we consider some more general term structure models and show how the market prices of bonds can be used as instrumental variables in empirical tests of the theory. Section 7 presents some models which include the effects of random inflation. In Section 8, we give some brief concluding comments.

2. THE UNDERLYING EQUILIBRIUM MODEL

In this section, we briefly review and specialize the general equilibrium model of Cox, Ingersoll, and Ross [**6**]. The model is a complete intertemporal description of a continuous time competitive economy. We recall that in this economy there is a single good and all values are measured in terms of units of this good. Production opportunities consist of a set of n linear activities. The vector of expected rates of return on these activities is α, and the covariance matrix of the rates of return is GG'. The components of α and G are functions of a k-dimensional vector Y which represents the state of the technology and is itself changing randomly over time. The development of Y thus determines the production opportunities that will be available to the economy in the future. The vector of expected changes in Y is μ and the covariance matrix of the changes is SS'.

The economy is composed of identical individuals, each of whom seeks to maximize an objective function of the form

$$(1) \qquad E\int_t^{t'} U(C(s), Y(s), s)\, ds,$$

where $C(s)$ is the consumption flow at time s, U is a Von Neumann-Morgenstern utility function, and t' is the terminal date. In performing this maximization, each individual chooses his optimal consumption C^*, the optimal proportion a^* of wealth W to be invested in each of the production processes, and the optimal proportion b^* of wealth to be invested in each of the contingent claims. These contingent claims are endogenously created securities whose payoffs are functions of W and Y. The remaining wealth to be invested in borrowing or lending at the interest rate r is then determined by the budget constraint. The indirect utility function J is determined by the solution to the maximization problem.

In equilibrium in this homogeneous society, the interest rate and the expected rates of return on the contingent claims must adjust until all wealth is invested in the physical production processes. This investment can be done either directly by individuals or indirectly by firms. Consequently, the equilibrium value of J is given by the solution to a planning problem with only the physical production processes available. For future reference, we note that the optimality conditions for the proportions invested will then have the form

$$\Psi \equiv \alpha W J_W + GG' a^* W^2 J_{WW} + GS' W J_{WY} - \lambda^* 1 \leq 0 \tag{2}$$

and $a^{*\prime}\Psi = 0$, where subscripts on J denote partial derivatives, J_{WY} is a $(k \times 1)$ vector whose ith element is J_{WY_i}, 1 is a $(k \times 1)$ unit vector, and λ^* is a Lagrangian multiplier. With J explicitly determined, the similar optimality conditions for the problem with contingent claims and borrowing and lending can be combined with the market clearing conditions to give the equilibrium interest rate and expected rates of return on contingent claims.

We now cite two principal results from [**6**] which we will need frequently in this paper. First, the equilibrium interest rate can be written explicitly as

$$\begin{aligned} r(W, Y, t) &= \frac{\lambda^*}{WJ_W} = a^{*\prime}\alpha + a^{*\prime}GG'a^* W\left(\frac{J_{WW}}{J_W}\right) + a^{*\prime}GS'\left(\frac{J_{WY}}{J_W}\right) \\ &= a^{*\prime}\alpha - \left(\frac{-J_{WW}}{J_W}\right)\left(\frac{\text{var } W}{W}\right) - \sum_{i=1}^{k}\left(\frac{-J_{WY_i}}{J_W}\right)\left(\frac{\text{cov } W, Y_i}{W}\right), \end{aligned} \tag{3}$$

where (cov W, Y_i) is the covariance of the changes in optimally invested wealth with the changes in the state variable Y_i, with (var W) and (cov Y_i, Y_j) defined in an analogous way; note that $a^{*\prime}\alpha$ is the expected rate of return on optimally invested wealth. Second, the equilibrium value of any contingent claim, F, must satisfy the following differential equation:

$$\begin{aligned} &\tfrac{1}{2}a^{*\prime}GG'a^* W^2 F_{WW} + a^{*\prime}GS' W F_{WY} + \tfrac{1}{2}\operatorname{tr}(SS'F_{YY}) \\ &\quad + (a^{*\prime}\alpha W - C^*)F_W + \mu' F_Y + F_t + \delta - rF \\ &= \phi_W F_W + \phi_Y F_Y, \end{aligned} \tag{4}$$

where $\delta(W, Y, t)$ is the payout flow received by the security and

$$\begin{aligned} \phi_W &= (a^{*\prime}\alpha - r)W, \\ \phi_Y &= \left(\frac{-J_{WW}}{J_W}\right) a^{*\prime}GS'W + \left(\frac{-J_{WY}}{J_W}\right)' SS'. \end{aligned} \tag{5}$$

In (4) subscripts on F denote partial derivatives; F_Y and F_{WY} are $(k \times 1)$ vectors and F_{YY} is a $(k \times k)$ matrix. The left hand side of (4) gives the excess expected return on the security over and above the risk free return, while the right hand side gives the risk premium that the security must command in equilibrium. For future reference, we note that (4) can be written in the alternative form:

$$\begin{aligned} &\tfrac{1}{2}(\text{var } W)F_{WW} + \sum_{i=1}^{k}(\text{cov } W, Y_i)F_{WY_i} + \tfrac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}(\text{cov } Y_i, Y_j)F_{Y_iY_j} \\ &\quad + [rW - C^*]F_W + \sum_{i=1}^{k}\left[\mu_i - \left(\frac{-J_{WW}}{J_W}\right)(\text{cov } W, Y_i)\right. \\ &\quad \left. - \sum_{j=1}^{k}\left(\frac{-J_{WY_j}}{J_W}\right)(\text{cov } Y_i, Y_j)\right]F_{Y_i} + F_t - rF + \delta = 0. \end{aligned} \tag{6}$$

To apply these formulas to the problem of the term structure of interest rates, we specialize the preference structure first to the case of constant relative risk aversion utility functions and then further to the logarithmic utility function. In particular, we let $U(C(s), Y(s), s)$ be independent of the state variable Y and have the form

$$(7)\qquad U(C(s), s) = e^{-\rho s}\left[\frac{C(s)^{\gamma}-1}{\gamma}\right],$$

where ρ is a constant discount factor.

It is easy to show that in this case the indirect utility function takes the form:[4]

$$(8)\qquad J(W, Y, t) = f(Y, t)U(W, t) + g(Y, t).$$

This special form brings about two important simplifications. First, the coefficient of relative risk aversion of the indirect utility function is constant, independent of both wealth and the state variables:

$$(9)\qquad \frac{-WJ_{WW}}{J_W} = 1-\gamma.$$

Second, the elasticity of the marginal utility of wealth with respect to each of the state variables does not depend on wealth, and we have

$$(10)\qquad \frac{-J_{WY}}{J_W} = \frac{-f_Y}{f}.$$

Furthermore, it is straightforward to verify that the optimal portfolio proportions a^* will depend on Y but not on W. Consequently, the vector of factor risk premiums, ϕ_Y, reduces to $(1-\gamma)a^{*\prime}GS' + (f_Y/f)SS'$, which depends only on Y. In addition, it can be seen from (3) that the equilibrium interest rate also depends only on Y.

The logarithmic utility function corresponds to the special case of $\gamma = 0$. For this case, it can be shown that $f(Y, t) = [1-\exp[-\rho(t'-t)]]/\rho$. The state-dependence of the indirect utility function thus enters only through $g(Y, t)$. As a result, ϕ_Y reduces further to $a^{*\prime}GS$. In addition, the particular form of the indirect utility function allows us to solve (2) explicitly for a^* as

$$(11)\qquad a^* = (GG')^{-1}\alpha + \left(\frac{1-1'(GG')^{-1}\alpha}{1'(GG')^{-1}1}\right)(GG')^{-1}1$$

when all production processes are active, with an analogous solution holding when some processes are inactive.

In the remainder of the paper, we will be valuing securities whose contractual terms do not depend explicitly on wealth. Since with constant relative risk aversion neither the interest rate r nor the factor risk premiums ϕ_Y depend on wealth, for such securities the partial derivatives F_W, F_{WW}, and F_{WY} are all equal to zero and the corresponding terms drop out of the valuation equation (4).

[4] This type of separability has been shown in other contexts by Hakansson [15], Merton [22], and Samuelson [32].

By combining these specializations, we find that the valuation equation (4) then reduces to

$$\frac{1}{2}\operatorname{tr}(SS'F_{YY})+[\mu'-a^{*\prime}GS']F_Y+F_t+\delta-rF=0. \tag{12}$$

Equation (12) will be the central valuation equation for this paper. We will use it together with various specifications about technological change to examine the implied term structure of interest rates.

3. A SINGLE FACTOR MODEL OF THE TERM STRUCTURE

In our first model of the term structure of interest rates, we assume that the state of technology can be represented by a single sufficient statistic or state variable. This is our most basic model, and we will examine it in some detail. This will serve to illustrate how a similarly detailed analysis can be conducted for the more complicated models that follow in Sections 5 and 6.

We make the following assumptions:

ASSUMPTION 1: *The change in production opportunities over time is described by a single state variable,* $Y(\equiv Y_1)$.

ASSUMPTION 2: *The means and variances of the rates of return on the production processes are proportional to Y.*[5] *In this way, neither the means nor the variances will dominate the portfolio decision for large values of Y. The state variable Y can be thought of as determining the rate of evolution of the capital stock in the following sense. If we compare a situation where* $Y=\bar{Y}$, *a constant, with a situation in which* $Y=2\bar{Y}$, *then the first situation has the same distribution of rate of return on a fixed investment in any process over a two-year period that the second situation has over a one-year period. We assume that the elements of* α *and* G *are such that the elements of* a^* *given by* (11) *are positive, so that all processes are always active, and that* $1'(GG')^{-1}\alpha$ *is greater than one.*[6]

ASSUMPTION 3: *The development of the state variable Y is given by the stochastic differential equation*

$$dY(t)=[\xi Y+\zeta]\,dt+v\sqrt{Y}\,dw(t), \tag{13}$$

where ξ *and* ζ *are constants, with* $\zeta\geqslant 0$, *and* v *is a* $1\times(n+k)$ *vector, each of whose components is the constant* v_0.

[5] Although our assumptions in this section do not satisfy all of the technical growth restrictions placed on the utility function and the coefficients of the production function in [6], they do in combination lead to a well-posed problem having an optimal solution with many useful properties. The optimal consumption function is $C^*(W,Y,t)=[\rho/(1-\exp(-\rho(t'-t))]W$ and the indirect utility function has the form $J(W,Y,t)=a(t)\log W+b(t)Y+c(t)$, where $a(t)$, $b(t)$, and $c(t)$ are explicitly determinable functions of time.

[6] The condition $1'(GG')^{-1}\alpha>1$, together with (13) and (14), insures that the interest rate will always be nonnegative. If $1'(GG')^{-1}\alpha<1$, the interest rate will always be nonpositive.

This structure makes it convenient to introduce the notation $\alpha \equiv \hat{\alpha} Y$, $GG' \equiv \Omega Y$, and $GS' \equiv \Sigma Y$, where the elements of $\hat{\alpha}$, Ω, and Σ are constants.

With these assumptions about technological change and our earlier assumptions about preferences, we can use (3) to write the equilibrium interest rate as

$$(14) \qquad r(Y) = \left(\frac{1'\Omega^{-1}\hat{\alpha} - 1}{1'\Omega^{-1}1} \right) Y.$$

The interest rate thus follows a diffusion process with

$$(15) \qquad \text{drift } r = \left(\frac{1'\Omega^{-1}\hat{\alpha} - 1}{1'\Omega^{-1}1} \right)(\xi Y + \zeta) \equiv \kappa(\theta - r),$$

$$\text{var } r = \left(\frac{1'\Omega^{-1}\hat{\alpha} - 1}{1'\Omega^{-1}1} \right)^2 \upsilon\upsilon' Y \equiv \sigma^2 r,$$

where κ, θ, and σ^2 are constants, with $\kappa\theta \geqslant 0$ and $\sigma^2 > 0$. It is convenient to define a new one-dimensional Wiener process, $z_1(t)$, such that:

$$(16) \qquad \sigma\sqrt{r}\, dz_1(t) \equiv \upsilon\sqrt{Y}\, dw(t);$$

this is permissible since each component of $w(t)$ is a Wiener process. The interest rate dynamics can then be expressed as:

$$(17) \qquad dr = \kappa(\theta - r)\, dt + \sigma\sqrt{r}\, dz_1.$$

For κ, $\theta > 0$, this corresponds to a continuous time first-order autoregressive process where the randomly moving interest rate is elastically pulled toward a central location or long-term value, θ. The parameter κ determines the speed of adjustment.[7]

An examination of the boundary classification criteria shows that r can reach zero if $\sigma^2 > 2\kappa\theta$. If $2\kappa\theta \geqslant \sigma^2$, the upward drift is sufficiently large to make the origin inaccessible.[8] In either case, the singularity of the diffusion coefficient at the origin implies that an initially nonnegative interest rate can never subsequently become negative.

The interest rate behavior implied by this structure thus has the following empirically relevant properties: (i) Negative interest rates are precluded. (ii) If the interest rate reaches zero, it can subsequently become positive. (iii) The absolute variance of the interest rate increases when the interest rate itself increases. (iv) There is a steady state distribution for the interest rate.

The probability density of the interest rate at time s, conditional on its value at the current time, t, is given by:

$$(18) \qquad f(r(s), s; r(t), t) = c\, e^{-u-\upsilon} \left(\frac{\upsilon}{u} \right)^{q/2} I_q(2(u\upsilon)^{1/2}),$$

[7] The discrete time equivalent of this model was tested by Wood [**38**], although, being concerned only with expectations, he left the error term unspecified.

[8] See Feller [**12**].

where

$$c \equiv \frac{2\kappa}{\sigma^2(1-e^{-\kappa(s-t)})},$$

$$u \equiv cr(t)\, e^{-\kappa(s-t)},$$

$$v \equiv cr(s),$$

$$q \equiv \frac{2\kappa\theta}{\sigma^2} - 1,$$

and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q. The distribution function is the noncentral chi-square, $\chi^2[2cr(s); 2q+2, 2u]$, with $2q+2$ degrees of freedom and parameter of noncentrality $2u$ proportional to the current spot rate.[9]

Straightforward calculations give the expected value and variance of $r(s)$ as:

$$E(r(s)|r(t)) = r(t)\, e^{-\kappa(s-t)} + \theta(1-e^{-\kappa(s-t)}),$$

$$\text{var}\,(r(s)|r(t)) = r(t)\left(\frac{\sigma^2}{\kappa}\right)(e^{-\kappa(s-t)} - e^{-2\kappa(s-t)}) + \theta\left(\frac{\sigma^2}{2\kappa}\right)(1-e^{-\kappa(s-t)})^2. \tag{19}$$

The properties of the distribution of the future interest rates are those expected. As κ approaches infinity, the mean goes to θ and the variance to zero, while as κ approaches zero, the conditional mean goes to the current interest rate and the variance to $\sigma^2 r(t)\cdot(s-t)$.

If the interest rate does display mean reversion ($\kappa, \theta > 0$), then as s becomes large its distribution will approach a gamma distribution. The steady state density function is:

$$f[r(\infty), \infty; r(t), t] = \frac{\omega^\nu}{\Gamma(\nu)} r^{\nu-1}\, e^{-\omega r}, \tag{20}$$

where $\omega \equiv 2\kappa/\sigma^2$ and $\nu \equiv 2\kappa\theta/\sigma^2$. The steady state mean and variance are θ and $\sigma^2\theta/2\kappa$, respectively.

Consider now the problem of valuing a default-free discount bond promising to pay one unit at time T.[10] The prices of these bonds for all T will completely determine the term structure. Under our assumptions, the factor risk premium in (12) is

$$\left[\hat{\alpha}'\Omega^{-1}\Sigma + \left(\frac{1-1'\Omega^{-1}\hat{\alpha}}{1'\Omega^{-1}1}\right)1'\Omega^{-1}\Sigma\right]Y \equiv \lambda Y. \tag{21}$$

[9] Processes similar to (17) have been extensively studied by Feller. The Laplace transform of (18) is given in Feller [12]. See Johnson and Kotz [17] for a description of the noncentral chi-square distribution. Oliver [27] contains properties of the modified Bessel function.

[10] A number of contractual provisions are sufficient to preclude default risk and make the value of a bond independent of the wealth of its seller. For example, the terms of the bond could specify that the seller must repurchase the bond at the price schedule given by (23) whenever his wealth falls to a designated level.

By using (15) and (21), we can write the fundamental equation for the price of a discount bond, P, most conveniently as

$$\tfrac{1}{2}\sigma^2 r P_{rr} + \kappa(\theta - r)P_r + P_t - \lambda r P_r - rP = 0, \tag{22}$$

with the boundary condition $P(r, T, T) = 1$. The first three terms in (22) are, from Ito's formula, the expected price change for the bond. Thus, the expected rate of return on the bond is $r + (\lambda r P_r / P)$. The instantaneous return premium on a bond is proportional to its interest elasticity. The factor λr is the covariance of changes in the interest rate with percentage changes in optimally invested wealth (the "market portfolio"). Since $P_r < 0$, positive premiums will arise if this covariance is negative ($\lambda < 0$).

We may note from (22) that bond prices depend on only one random variable, the spot interest rate, which serves as an instrumental variable for the underlying technological uncertainty. While the proposition that current (and future) interest rates play an important, and to a first approximation, predominant role in determining the term structure would meet with general approval, we have seen that this will be precisely true only under special conditions.[11]

By taking the relevant expectation (see Cox, Ingersoll, and Ross [**6**]), we obtain the bond prices as:

$$P(r, t, T) = A(t, T)\, e^{-B(t,T)r},$$

where

$$A(t, T) \equiv \left[\frac{2\gamma e^{[(\kappa+\lambda+\gamma)(T-t)]/2}}{(\gamma+\kappa+\lambda)(e^{\gamma(T-t)}-1)+2\gamma}\right]^{2\kappa\theta/\sigma^2},$$

$$B(t, T) \equiv \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+\kappa+\lambda)(e^{\gamma(T-t)}-1)+2\gamma}, \tag{23}$$

$$\gamma \equiv ((\kappa+\lambda)^2 + 2\sigma^2)^{1/2}.$$

The bond price is a decreasing convex function of the interest rate and an increasing (decreasing) function of time (maturity). The parameters of the interest rate process have the following effects. The bond price is a decreasing convex function of the mean interest rate level θ and an increasing concave (decreasing convex) function of the speed of adjustment parameter κ if the interest rate is greater (less) than θ. Both of these results are immediately obvious from their effects on expected future interest rates. Bond prices are an increasing concave function of the "market" risk parameter λ. Intuitively, this is mainly because higher values of λ indicate a greater covariance of the interest rate with wealth. Thus, with large λ it is more likely that bond prices will be higher when wealth is low and, hence, has greater marginal utility. The bond price is an increasing

[11] In our framework, the most important circumstances sufficient for bond prices to depend only on the spot interest rate are: (i) individuals have constant relative risk aversion, uncertainty in the technology can be described by a single variable, and the interest rate is a monotonic function of this variable, or (ii) changes in the technology are nonstochastic and the interest rate is a monotonic function of wealth.

concave function of the interest rate variance σ^2. Here several effects are involved. The most important is that a larger σ^2 value indicates more uncertainty about future real production opportunities, and thus more uncertainty about future consumption. In such a world, risk-averse investors would value the guaranteed claim in a bond more highly.

The dynamics of bond prices are given by the stochastic differential equation:

$$dP = r[1-\lambda B(t,T)]P\,dt - B(t,T)P\sigma\sqrt{r}\,dz_1. \tag{24}$$

For this single state variable model, the returns on bonds are perfectly negatively correlated with changes in the interest rate. The returns are less variable when the interest rate is low. Indeed, they become certain if a zero interest rate is reached, since interest rate changes are then certain. As we would intuitively expect, other things remaining equal, the variability of returns decreases as the bond approaches maturity. In fact, letting t approach T and denoting $T-t$ as Δt, we find that the expected rate of return is $r\Delta t + O(\Delta t^2)$ and the variance of the rate of return is $O(\Delta t^2)$ rather than $O(\Delta t)$, as would be the case for the returns on an investment in the production processes over a small interval. It is in this sense that the return on very short-term bonds becomes certain.

Bonds are commonly quoted in terms of yields rather than prices. For the discount bonds we are now considering, the yield-to-maturity, $R(r,t,T)$, is defined by $\exp[-(T-t)R(r,t,T)] \equiv P(r,t,T)$. Thus, we have:

$$R(r,t,T) = [rB(t,T) - \log A(t,T)]/(T-t). \tag{25}$$

As maturity nears, the yield-to-maturity approaches the current interest rate independently of any of the parameters. As we consider longer and longer maturities, the yield approaches a limit which is independent of the current interest rate:

$$R(r,t,\infty) = \frac{2\kappa\theta}{\gamma+\kappa+\lambda}. \tag{26}$$

When the spot rate is below this long-term yield, the term structure is uniformly rising. With an interest rate in excess of $\kappa\theta/(\kappa+\lambda)$, the term structure is falling. For intermediate values of the interest rate, the yield curve is humped.

Other comparative statics for the yield curve are easily obtained from those of the bond pricing function. An increase in the current interest rate increases yields for all maturities, but the effect is greater for shorter maturities. Similarly, an increase in the steady state mean θ increases all yields, but here the effect is greater for longer maturities. The yields to maturity decrease as σ^2 or λ increases, while the effect of a change in κ may be of either sign depending on the current interest rate.

There has always been considerable concern with unbiased predictions of future interest rates. In the present situation, we could work directly with equation (19), which gives expected values of future interest rates in terms of the current rate and the parameters κ and θ. However, in the rational expectations model we have constructed, all of the information that is currently known about the future movement of interest rates is impounded in current bond prices and the term structure. If the model is correct, then any single parameter can be determined from the term structure and the values of the other parameters.

This approach is particularly important when the model is extended to allow a time-dependent drift term, $\theta(t)$. We can then use information contained in the term structure to obtain $\theta(t)$ and expected future spot rates without having to place prior restrictions on its functional form.

Now, the future expected spot rate given by (19) is altered to:

$$E(r(T)|r(t)) = r(t)\, e^{-\kappa(T-t)} + \kappa \int_t^T \theta(s)\, e^{-\kappa(T-s)}\, ds. \tag{27}$$

The bond pricing formula (30), in turn, is modified to:

$$P(r, t, T) = \hat{A}(t, T)\, e^{-B(t,T)r}, \tag{28}$$

where

$$\hat{A}(t, T) = \exp\left(-\kappa \int_t^T \theta(s) B(s, T)\, ds\right), \tag{29}$$

which reduces to (23) when $\theta(s)$ is constant.

Assuming, for illustration, that the other process parameters are known, we can then use the term structure to determine unbiased forecasts of future interest rates. By (28), $\hat{A}(t, T)$ is an observable function of T, given the term structure and the known form of $B(t, T)$, and standard techniques can be invoked to invert (29) and obtain an expression for $\theta(t)$ in terms of $\hat{A}(t, T)$ and $B(t, T)$. Equation (27) can now be used to obtain predictions of the expected values of future spot rates implicit in the current term structure.

Note that these are not the same values that would be given by the traditional expectations assumption that the expected values of future spot rates are contained in the term structure in the form of implicit forward rates. In a continuous-time model, the forward rate $\hat{r}(T)$ is given by $-P_T/P$. Then, by differentiating (28):

$$\begin{aligned} \hat{r}(T) &= -P_T(r, t, T)/P(r, t, T) \\ &= rB_T(t, T) + \kappa \int_t^T \theta(s) B_T(s, T)\, ds. \end{aligned} \tag{30}$$

Comparing (27) and (30), we see they have the same general form. However, the traditional forward rate predictor applies the improper weights $B_T(s, T) \neq e^{-\kappa(T-s)}$, resulting in a biased prediction.

A number of alternative specifications of time dependence may also be included with only minor changes in the model. One particularly tractable example leads to an interest rate of $\bar{r}(t) + g(t)$, where $\bar{r}(t)$ is given by (17) and $g(t)$ is a function which provides a positive lower bound for the interest rate. The essential point in all such cases is that in the rational expectations model, the current term structure embodies the information required to evaluate the market's probability distribution of the future course of interest rates. Furthermore, the term structure can be inverted to find these expectations.

Other single variable specifications of technological change will in turn imply other stochastic properties for the interest rate. It is easy to verify that in our model if α and GG' are proportional to some function $h(Y, t)$, then the interest rate will also be proportional to $h(Y, t)$. By a suitable choice of $h(Y, t)$, $\mu(Y, t)$, and $S(Y, t)$, a wide range of a priori properties of interest rate movements can be included within the context of a completely consistent model.

4. VALUING ASSETS WITH GENERAL INTEREST RATE DEPENDENT PAYOFFS

Our valuation framework can easily be applied to other securities whose payoffs depend on interest rates, such as options on bonds and futures on bonds. This flexibility enables the model to make predictions about the pricing patterns that should prevail simultaneously across several financial markets. Consequently, applications to other securities may permit richer and more powerful empirical tests than could be done with the bond market alone.

As an example of valuing other kinds of interest rate securities, consider options on bonds. Denote the value at time t of a call option on a discount bond of maturity date s, with exercise price K and expiration date T as $C(r, t, T; s, K)$.[12] The option price will follow the basic valuation equation with terminal condition:

$$C(r, t, T; s, K) = \max[P(r, T, s) - K, 0]. \tag{31}$$

It is understood that $s \geqslant T \geqslant t$, and K is restricted to be less than $A(T, s)$, the maximum possible bond price at time T, since otherwise the option would never be exercised and would be worthless. By again taking the relevant expectations, we arrive at the following formula for the option price:

$$\begin{aligned} C(r, t, T; s, K) \\ &= P(r, t, s)\chi^2\left(2r^*[\phi + \psi + B(T, s)]; \frac{4\kappa\theta}{\sigma^2}, \frac{2\phi^2 r e^{\gamma(T-t)}}{\phi + \psi + B(T, s)}\right) \\ &\quad - KP(r, t, T)\chi^2\left(2r^*[\phi + \psi]; \frac{4\kappa\theta}{\sigma^2}, \frac{2\phi^2 r e^{\gamma(T-t)}}{\phi + \psi}\right), \end{aligned} \tag{32}$$

where

$$\gamma \equiv ((\kappa + \lambda)^2 + 2\sigma^2)^{1/2},$$

$$\phi \equiv \frac{2\gamma}{\sigma^2(e^{\gamma(T-t)} - 1)},$$

$$\psi \equiv (\kappa + \lambda + \gamma)/\sigma^2,$$

$$r^* \equiv \left[\log\left(\frac{A(T, s)}{K}\right)\right] / B(T, s),$$

[12] Since the underlying security, a discount bond, makes no payments during the life of the option, the analysis of Merton [23] implies that premature exercise is never optimal, and, hence, American and European calls have the same value.

and $\chi^2(\cdot)$ is the previously introduced noncentral chi-square distribution function. r^* is the critical interest rate below which exercise will occur; i.e., $K = P(r^*, T, s)$.

The call option is an increasing function of maturity (when the expiration date on which the underlying bond matures remains fixed). Call options on stocks are increasing functions of the interest rate, partly because such an increase reduces the present value of the exercise price. However, here an increase in the interest rate will also depress the price of the underlying bond. Numerical analysis indicates that the latter effect is stronger and that the option value is a decreasing convex function of the interest rate. The remaining comparative statics are indeterminate.

5. A COMPARISON WITH BOND PRICING BY ARBITRAGE METHODS

In this section, we briefly compare our methodology to some alternative ways to model bond pricing in continuous time. It is useful to do this now rather than later because the model of Section 3 provides an ideal standard for comparison.

Our approach begins with a detailed description of the underlying economy. This allows us to specify the following ingredients of bond pricing: (a) the variables on which the bond price depends, (b) the stochastic properties of the underlying variables which are endogenously determined, and (c) the exact form of the factor risk premiums. In [**21**], Merton shows that if one begins instead by imposing assumptions directly about (a) and (b), then Ito's formula can be used to state the excess expected return on a bond in the same form as the left-hand side of (4). If the functional form of the right-hand side of (4) were known, then one could obtain a bond pricing equation. For example, if one arbitrarily assumed that bond prices depend only on the spot interest rate r, that the interest rate follows the process given by (17), and that the excess expected return on a bond with maturity date T is $Y(r, t, T)$, then one would obtain

$$(33) \qquad \tfrac{1}{2}\sigma^2 r P_{rr} + \kappa(\theta - r)P_r + P_t - rP = Y(r, t, T).$$

If there is some underlying equilibrium which will support the assumptions (a) and (b), then there must be some function Y for which bond prices are given by (33). However, as Merton notes, this derivation in itself provides no way to determine Y or to relate it to the underlying real variables.

An arbitrage approach to bond pricing was developed in a series of papers by Brennan and Schwartz [**3**], Dothan [**10**], Garman [**14**], Richard [**28**], and Vasicek [**37**]. Arguments similar to those employed in the proof of Theorem 2 of Cox, Ingersoll, and Ross [**6**] are used to show that if there are no arbitrage opportunities, Y must have the form

$$(34) \qquad Y(r, t, T) = \psi(r, t)P_r(r, t, T),$$

where ψ is a function depending only on calendar time and not on the maturity date of the bond. This places definite restrictions on the form of the excess expected return; not all functions Y will satisfy both (33) and (34).

There are some potential problems, however, in going one step further and using the arbitrage approach to determine a complete and specific model of the term structure. The approach itself provides no way of guaranteeing that there is some underlying equilibrium for which assumptions (a) and (b) are consistent. Setting this problem aside, another difficulty arises from the fact that the arbitrage approach does not imply that every choice of ψ in (34) will lead to bond prices which do not admit arbitrage opportunities. Indeed, closing the model by assuming a specific functional form for ψ can lead to internal inconsistencies.

As an example of the potential problem, consider (33) with Y as shown in (34). This gives the valuation equation

$$(35)\qquad \tfrac{1}{2}\sigma^2 rP_{rr} + \kappa(\theta - r)P_r + P_t - rP = \psi(r, t)P_r,$$

which is identical to (22) apart from a specification of the function ψ. We could now close the model by assuming that ψ is linear in the spot rate, $\psi(r, t) = \psi_0 + \lambda r$. The solution to (35) is then

$$(36)\qquad P(r, t, T) = [A(t, T)]^{(\kappa\theta - \psi_0)/\kappa\theta} \exp[-rB(t, T)],$$

and the dynamic behavior of the bond price is given by

$$(37)\qquad dP = [r - (\psi_0 + \lambda r)B(t, T)]P\,dt - B(t, T)\sigma\sqrt{r}P\,dz_1.$$

The linear form assumed for the risk premium seems quite reasonable and would appear to be a good choice for empirical work, but it in fact produces a model that is not viable. This is most easily seen when $r = 0$. In this case, the bond's return over the next instant is riskless; nevertheless, it is appreciating in price at the rate $-\psi_0 B(t, T)$, which is different from the prevailing zero rate of interest.[13] We thus have a model that guarantees arbitrage opportunities rather than precluding them. The difficulty, of course, is that there is no underlying equilibrium which would support the assumed premiums.

The equilibrium approach developed here thus has two important advantages over alternative methods of bond pricing in continuous time. First, it automatically insures that the model can be completely specified without losing internal consistency. Second, it provides a way to predict how changes in the underlying real economic variables will affect the term structure.

6. MULTIFACTOR TERM STRUCTURE MODELS AND THE USE OF PRICES AS INSTRUMENTAL VARIABLES

In Section 3, we specialized the general equilibrium framework of Cox, Ingersoll, and Ross [**6**] to develop a complete model of bond pricing. We purposely chose a simple specialization in order to illustrate the detailed information that such a model can produce. In the model, the prices of bonds of all maturities depended on a single random explanatory factor, the spot interest rate. Although the resulting term structure could assume several alternative shapes, it is inherent

[13] As stated earlier, the origin is accessible only if $\sigma^2 > 2\kappa\theta$. Somewhat more complex arguments can be used to demonstrate that the model is not viable even if the origin is inaccessible.

in a single factor model that price changes in bonds of all maturities are perfectly correlated. Such a model also implies that bond prices do not depend on the path followed by the spot rate in reaching its current level. For some applications, these properties may be too restrictive. However, more general specifications of technological opportunities will in turn imply more general bond pricing models. The resulting multifactor term structures will have more flexibility than the single factor model, but they will inevitably also be more cumbersome and more difficult to analyze.

To illustrate the possibilities, we consider two straightforward generalizations of our previous model. Suppose that in our description of technological change in (13) and (15), the central tendency parameter θ is itself allowed to vary randomly according to the equation

$$d\theta = \nu(Y - \theta)\, dt, \tag{38}$$

where ν is a positive constant. That is, we let $\theta \equiv Y_2$ and $\mu_2 = \nu(Y_1 - Y_2)$. The value of θ at any time will thus be an exponentially weighted integral of past values of Y. It can then be verified that the interest rate r is again given by (14) and that the bond price P will have the form

$$P(r, \theta, t, T) = \exp\left[-rf(t, T) - \theta g(t, T)\right], \tag{39}$$

where f and g are explicitly determinable functions of time. In this case, both the yields-to-maturity of discount bonds and the expected values of future spot rates are linear functions of current and past spot rates.[14]

As a second generalization, suppose that the production coefficients α and GG' are proportional to the sum of two independent random variables, Y_1 and Y_2, each of which follows an equation of the form (13). Then it can be shown that the spot interest rate r will be proportional to the sum of Y_1 and Y_2 and that bond prices will again have the exponential form

$$P(r, Y_2, t, T) = f(t, T) \exp\left[-rg(t, T) - Y_2 h(t, T)\right], \tag{40}$$

where f, g, and h are other explicitly determinable functions of time. In this model, price changes in bonds of all maturities are no longer perfectly correlated.

Each of these generalizations gives a two factor model of the term structure, and the resulting yield curves can assume a wide variety of shapes. Further multifactor generalizations can be constructed along the same lines.

In each of the models considered in this section, one of the explanatory variables is not directly observable. Multifactor generalizations will typically inherit this drawback to an even greater degree. Consequently, it may be very convenient for empirical applications to use some of the endogenously determined prices as instrumental variables to eliminate the variables that cannot be directly observed. In certain instances, it will be possible to do so. Let us choose the spot rate, r,

[14] Studies which have expressed expected future spot rates as linear combinations of current and past spot rates include Bierwag and Grove [2], Cagan [4], De Leeuw [8], Duesenberry [11], Malkiel [19], Meiselman [20], Modigliani and Shiller [24], Modigliani and Sutch [25], Van Horne [36], and Wood [38]. Cox, Ingersoll, and Ross [5] examine this issue in a diffusion setting.

and a vector of long interest rates, l, as instrumental variables. In general, each of these interest rates will be functions of W (unless the common utility function is isoelastic) and all the state variables. If it is possible to invert this system globally and express the latter as twice differentiable functions of r and l, then r and l can be used as instrumental variables in a manner consistent with the general equilibrium framework.

For the purposes of illustration, suppose that there are two state variables, Y_1 and Y_2, and that utility is isoelastic so that the level of wealth is immaterial. Then, for instrumental variables r and l, a scalar, direct but involved calculations show that the valuation equation (4) may be rewritten as:

$$\text{(41)} \qquad \tfrac{1}{2}(\operatorname{var} r)F_{rr} + (\operatorname{cov} r, l)F_{rl} + \tfrac{1}{2}(\operatorname{var} l)F_{ll} + [\mu_r - \lambda_r(r, l)]F_r + [\mu_l - \lambda_l(r, l)]F_l - rF + F_t + \delta = 0.$$

The functions λ_r and λ_l serve the role of the factor risk premiums in (5). They are related to the factor risk premiums, ϕ_Y, by:

$$\lambda_r(r, l) = \left[\psi_1 \frac{\partial g}{\partial l} - \psi_2 \frac{\partial f}{\partial l}\right] \Big/ \Delta,$$

$$\lambda_l(r, l) = \left[\psi_2 \frac{\partial f}{\partial r} - \psi_1 \frac{\partial g}{\partial r}\right] \Big/ \Delta,$$

where

$$\text{(42)} \qquad Y_1 \equiv f(r, l, t), \qquad Y_2 \equiv g(r, l, t),$$
$$\phi_{Y_1}(Y_1, Y_2, t) \equiv \psi_1(r, l, t), \qquad \phi_{Y_2}(Y_1, Y_2, t) \equiv \psi_2(r, l, t),$$

and

$$\Delta \equiv \frac{\partial f}{\partial r}\frac{\partial g}{\partial l} - \frac{\partial f}{\partial l}\frac{\partial g}{\partial r}.$$

Thus far we have not used the fact that l is an interest rate, and the transformation of (4) to (41) can be performed for an arbitrary instrumental variable if the inversion is possible. The advantage of choosing an interest rate instrument is that the second risk factor premium λ_l and the drift μ_l can be eliminated from (41) as follows.

Let Q denote the value of the particular bond for which l is the continuously compounded yield-to-maturity. Denote the payment flow from the bond, including both coupons and return of principal, by $c(t)$. In general, this flow will be zero most of the time, with impulses representing an infinite flow rate when payments are made. Since by definition $Q \equiv \int_t^T c(s) \exp[-l(s-t)]\, ds$, we can write:

$$\text{(43)} \qquad Q \equiv A_0(l), \quad Q_l = A_1(l),$$
$$Q_{ll} = A_2(l), \quad Q_t = -c(t) + lA_0(l) = -\delta + lA_0(l),$$
$$Q_r = Q_{rr} = Q_{rl} = 0,$$

where

$$A_n \equiv \int_t^T (t-s)^n c(s)\, e^{-l(s-t)}\, ds,$$

and the integral is to be interpreted in the Stieltjes sense. If (43) is substituted into (41), we then obtain:

$$(44) \qquad \mu_l - \lambda_l(r, l) = \frac{(r-l)A_0(l) - \frac{1}{2}(\text{var } l)A_2(l)}{A_1(l)},$$

and the unobservable factor risk premium may be replaced by the observable function in (44). If Q is a consol bond with coupons paid continuously at the rate c, then $A_0 = c/l$, $A_1 = -c/l^2$, $A_2 = 2c/l^3$, and (44) may be written as:[15]

$$(45) \qquad \mu_l - \lambda_l(r, l) = \frac{(\text{var } l)}{l} + l(l-r).$$

These representations may be a useful starting point for empirical work. However, it is important to remember that they cannot be fully justified without considering the characteristics of the underlying economy. In the next section, we examine some additional multiple state variable models, all of which could be reexpressed in this form.

7. UNCERTAIN INFLATION AND THE PRICING OF NOMINAL BONDS

The model presented here deals with a real economy in which money would serve no purpose. To provide a valid role for money, we would have to introduce additional features which would lead far afield of our original intent. However, for a world in which changes in the money supply have no real effects, we can introduce some aspects of money and inflation in an artificial way by imagining that one of the state variables represents a price level and that some contracts have payoffs whose real value depends on this price level. That is, they are specified in nominal terms. None of this requires any changes in the general theory.

Suppose that we let the price level, p, be the kth state variable. Since we assume that this variable has no effect on the underlying real equilibrium, the functions α, μ, G, S, and J will not depend on p. Of course, this would not preclude changes in p from being statistically correlated with changes in real wealth and the other state variables. Under these circumstances, the real value of a claim whose payoff is specified in nominal terms still satisfies equation (4). All that needs to be done is to express the nominal payoff in real terms for the boundary conditions. Alternatively, the valuation equation (4) will also still hold if p is a differentiable function of W, Y, and t.[16]

[15] See Brennan and Schwartz [3] for this representation.

[16] If one wished to make real money balances an argument in the direct utility function U, it would be straightforward to do so in our model. A utility-maximizing money supply policy would depend only on the state variables, real wealth, and time, so the induced price level would depend only on these variables as well.

We can illustrate some of these points in the context of the model of Section 3. Let us take a second state variable to be the price level, $p(\equiv Y_2)$, and consider how to value a contract which will at time T pay with certainty an amount $1/p(T)$. Call this a nominal unit discount bond, and denote its value at time t in real terms as $N(r, p, t, T)$. Suppose that the price level p moves according to

$$dp = \mu(p)\, dt + \sigma(p)\, dw_{n+2}(t) \tag{46}$$

and that it is uncorrelated with W and Y_1. Assume also that the coefficients in (45) are such that $E[p^{-1}(s)]$ exists for all finite s.

We would then have the valuation equation for N

$$\tfrac{1}{2}\sigma^2 r N_{rr} + \tfrac{1}{2}\sigma^2(p) N_{pp} + [\kappa\theta - (\kappa+\lambda)r]N_r + \mu(p) N_p + N_t - rN = 0 \tag{47}$$

with terminal condition $N(r, p, T, T) = 1/p(T)$. It can be directly verified that the solution is

$$N(r, p, t, T) = P(r, t, T) \underset{p(t),t}{E} [1/p(T)] \tag{48}$$

where P is the price of a real discount bond given in (23).

In this formulation, the expected inflation rate changes only with the price level. For the commonly assumed case of lognormally distributed prices, however, $\mu(p) = \mu_p p$, $\sigma(p) = \sigma_p p$, and

$$N(r, p, t, T) = e^{-(\mu_p - \sigma_p^2)(T-t)} P(r, t, T)/p(t), \tag{49}$$

so in this case the price of a nominal bond in nominal terms, $\hat{N} \equiv p(t)N$, would be independent of the current price level. With lognormally distributed prices, the expected inflation rate is constant, although of course realized inflation will not be.

As a somewhat more general example, we can separate the expected inflation rate factor from the price level factor and identify it with a third state variable. Again no change in the general theory is necessary. Label the expected inflation rate as y. We propose two alternative models for the behavior of the inflation rate: (i) Model 1,

$$dy = \kappa_1 y(\theta_1 - y)\, dt + \sigma_1 y^{3/2}\, dz_3; \tag{50}$$

(ii) Model 2,

$$dy = \kappa_2(\theta_2 - y)\, dt + \sigma_2 y^{1/2}\, dz_3 \tag{51}$$

with the stochastic differential equation governing the movement of the price level being in each case

$$dp = yp\, dt + \sigma_p p y^{1/2}\, dz_2 \tag{52}$$

with $(\text{cov } y, p) \equiv \rho\sigma_1\sigma_p y^2 p$ in Model 1, $(\text{cov } y, p) \equiv \rho\sigma_1\sigma_p y p$ in Model 2, and $\sigma_p < 1$. Here, as in (17), we have for convenience defined $z_2(t)$ and $z_3(t)$ as the appropriate linear combinations of $w_{n+2}(t)$ and $w_{n+3}(t)$.

Model 1 may well be the better choice empirically, since informal evidence suggests that the relative (percentage) variance of the expected inflation rate increases as its level increases. Model 1 has this property, while Model 2 does not. However, the solution to Model 2 is more tractable, so we will record both for possible empirical use. In both models the expected inflation rate is pulled toward a long-run equilibrium level. Both models also allow for correlation between changes in the inflation rate and changes in the price level, thus allowing for positive or negative extrapolative forces in the movement of the price level.

The valuation equation for the real value of a nominal bond, specialized for our example with Model 1, will then be

$$\begin{aligned}(53)\qquad &\tfrac{1}{2}\sigma^2 rN_{rr}+\tfrac{1}{2}\sigma_1^2 y^3 N_{yy}+\rho\sigma_1\sigma_p y^2 pN_{yp}+\tfrac{1}{2}\sigma_p^2 p^2 yN_{pp}+[\kappa\theta-(\kappa+\lambda)r]N_r\\ &+\kappa_1 y(\theta_1-y)N_y+ypN_p+N_t-rN=0\end{aligned}$$

with $N(r, y, p, T, T)=1/p(T)$. The solution to equation (53) is

$$N(r, y, p, t, T)=\frac{\Gamma(\nu-\delta)}{\Gamma(\nu)}\left[\frac{c(t)}{y}\right]^{\delta} M\left(\delta, \nu, -\frac{c(t)}{y}\right)P(r, t, T)/p(t),$$

where

$$\begin{aligned}&c(t)\equiv\frac{2\kappa_1\theta_1}{\sigma_1^2(e^{\kappa_1\theta_1(T-t)}-1)},\\ (54)\qquad &\delta\equiv[[(\kappa_1+\rho\sigma_1\sigma_p+\tfrac{1}{2}\sigma_1^2)^2+2(1-\sigma_p^2)\sigma_1^2]^{1/2}-(\kappa_1+\rho\sigma_1\sigma_p+\tfrac{1}{2}\sigma_1^2)]/\sigma_1^2,\\ &\nu\equiv 2[(1+\delta)\sigma_1^2+\kappa_1+\rho\sigma_1\sigma_p]/\sigma_1^2,\end{aligned}$$

$M(\cdot, \cdot, \cdot)$ is the confluent hypergeometric function, and $\Gamma(\cdot)$ is the gamma function.[17]

Proceeding in the same way with Model 2, we obtain the valuation equation:

$$\begin{aligned}(55)\qquad &\tfrac{1}{2}\sigma^2 rN_{rr}+\tfrac{1}{2}\sigma_2^2 yN_{yy}+\rho\sigma_2\sigma_p ypN_{yp}+\tfrac{1}{2}\sigma_p^2 yp^2 N_{pp}+[\kappa\theta-(\kappa+\lambda)r]N_r\\ &+\kappa_2[\theta_2-y]N_y+ypN_p+N_t-rN=0\end{aligned}$$

with $N(r, y, p, T, T)=1/p(T)$. The corresponding valuation formula is:

$$\begin{aligned}(56)\qquad &N(r, y, p, t, T)\\ &=\left(\frac{2\xi e^{[(\kappa_2+\rho\sigma_2\sigma_p+\xi)(T-t)]/2}}{(\xi+\kappa_2+\rho\sigma_2\sigma_p)(e^{\xi(T-t)}-1)+2\xi}\right)^{2\kappa_2\theta_2/\sigma_2^2}\\ &\quad\times\exp\left(\frac{-2(e^{\xi(T-t)}-1)(1-\sigma_p^2)y}{(\xi+\kappa_2+\rho\sigma_2\sigma_p)(e^{\xi(T-t)}-1)+2\xi}\right)P(r, t, T)/p(t),\end{aligned}$$

where

$$\xi\equiv[(\kappa_2+\rho\sigma_2\sigma_p)^2+2\sigma_2^2(1-\sigma_p^2)]^{1/2}.$$

[17] Slater [34] gives properties of the confluent hypergeometric function.

The term structure of interest rates implied by (54) and (56) can assume a wide variety of shapes, depending on the relative values of the variables and parameters. More complex models incorporating more detailed effects can be built along the same lines.

Throughout our paper, we have used specializations of the fundamental valuation equation (6). This equation determines the real value of a contingent claim as a function of real wealth and the state variables. For some empirical purposes, it may be convenient to have a corresponding valuation equation in which all values are expressed in nominal terms.

In our setting, this is given by the following proposition. In this proposition, we let nominal wealth be $X \equiv pW$, the indirect utility function in terms of nominal wealth be $V(X, Y, t) \equiv J(X/p, Y, t) \equiv J(W, Y, t)$, and the nominal value of a claim in terms of nominal wealth be $H(X, Y, t) \equiv pF(X/p, Y, t) \equiv pF(W, Y, t)$. As before, we let p be the kth element of Y.

PROPOSITION: *The nominal value of a contingent claim in terms of nominal wealth, $H(X, Y, t)$, satisfies the partial differential equation*

$$(57) \qquad \tfrac{1}{2}(\operatorname{var} X)H_{XX} + \sum_{i=1}^{k} (\operatorname{cov} X, Y_i)H_{XY_i} + \tfrac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} (\operatorname{cov} Y_i, Y_j)H_{Y_iY_j}$$
$$+ (\iota X - pC^*)H_x + \sum_{i=1}^{k}\left[\mu_i - \left(\frac{-V_{XX}}{V_X}\right)(\operatorname{cov} X, Y_i)\right.$$
$$\left. - \sum_{j=1}^{k}\left(\frac{-V_{XY_j}}{V_X}\right)(\operatorname{cov} Y_i, Y_j)\right]H_{Y_i} + H_t + p\delta - \iota H = 0,$$

where the nominal interest rate, ι, is given by

$$(58) \qquad \iota = \alpha_X - \left(\frac{-V_{XY}}{V_X}\right)\left(\frac{\operatorname{var} X}{X}\right) - \sum_{i=1}^{k}\left(\frac{-V_{XY_i}}{V_X}\right)\left(\frac{\operatorname{cov} X, Y_i}{X}\right)$$

and α_X is the expected rate of return on nominal wealth,

$$(59) \qquad \alpha_X = a^{*\prime}\alpha + \left(\frac{\mu_p}{p}\right) + \left(\frac{\operatorname{cov} p, X}{pX}\right) - \left(\frac{\operatorname{var} p}{p^2}\right).$$

PROOF: Ito's multiplication rule implies that

$$(\operatorname{var} W) = (1/p^2)(\operatorname{var} X) - (2X/p^3)(\operatorname{cov} X, p) + (X^2/p^4)(\operatorname{var} p),$$
$$(\operatorname{cov} W, p) = (1/p)(\operatorname{cov} X, p) - (X/p^2)(\operatorname{var} p),$$
$$(\operatorname{cov} W, Y) = (1/p)(\operatorname{cov} X, Y) - (X/p^2)(\operatorname{cov} p, Y),$$

and

$$\alpha_X = a^{*\prime}\alpha + (\mu_p/p) + (1/pX)(\operatorname{cov} X, p) - (1/p^2)(\operatorname{var} p).$$

With

$$J(W, Y, t) \equiv J(X/p, Y, t) \equiv V(X, Y, t),$$

we have

$$(J_{WW}/J_W) = p(V_{XX}/V_X),$$
$$(J_{WY_i}/J_W) = (V_{XY_i}/V_X), \qquad \text{and}$$
$$(V_{Xp}/V_X) = -(1/p) - (X/p)(V_{XX}/V_X).$$

Equation (57) follows by writing the derivatives of $F(W, Y, t)$ in terms of those of $H(X, Y, t)$ and substituting all of the above into (6). The nominal interest rate can then be identified as the nominal payout flow necessary to keep the nominal value of a security identically equal to one, which is ι as given in (58).

Q.E.D.

A comparison of (57) and (58) with (6) and (3) shows that the interest rate equation and the fundamental valuation equation have exactly the same form when all variables are expressed in nominal terms as when all variables are expressed in real terms. By using the arguments given in the proof of the proposition, the nominal interest rate can be expressed in terms of real wealth as

$$\text{(60)} \qquad \iota = r + \left(\frac{1}{p}\right)\left[\mu_p - \left(\frac{-J_{WW}}{J_W}\right)(\text{cov } W, p) - \sum_{i=1}^{k}\left(\frac{-J_{WY_i}}{J_W}\right)(\text{cov } Y_i, p) - \left(\frac{\text{var } p}{p}\right)\right],$$

where r, the real interest rate, is as given by equation (3). The term (μ_p/p) is the expected rate of inflation. The remaining terms may in general have either sign, so the nominal interest rate may be either greater or less than the sum of the real interest rate and the expected inflation rate.[18]

8. CONCLUDING COMMENTS

In this paper, we have applied a rational asset pricing model to study the term structure of interest rates. In this model, the current prices and stochastic properties of all contingent claims, including bonds, are derived endogenously. Anticipations, risk aversion, investment alternatives, and preferences about the timing of consumption all play a role in determining the term structure. The model thus includes the main factors traditionally mentioned in a way which is consistent with maximizing behavior and rational expectations.

By exploring specific examples, we have obtained simple closed form solutions for bond prices which depend on observable economic variables and can be tested. The combination of equilibrium intertemporal asset pricing principles and appropriate modelling of the underlying stochastic processes provides a powerful tool for deriving consistent and potentially refutable theories. This is the first

[18] For a related discussion, see Fischer [**13**].

such exercise along these lines, and the methods developed should have many applications beyond those which we considered here.

In a separate paper, Cox, Ingersoll, and Ross [5], we use our approach to examine some aspects of what may be called traditional theories of the term structure. There we show that some forms of the classical expectations hypothesis are consistent with our simple equilibrium model and more complex ones, while other forms in general are not. We also show the relationship between some continuous time equilibrium models and traditional theories which express expected future spot rates as linear combinations of past spot rates.

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